Continuous Markovian Logic - From Complete Axiomatization to the Metric Space of Formulas

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Motivation

Complex systems are often modelled as stochastic processes

biological and ecological systems, physical systems, social systems, financial systems

- to encapsulate a lack of knowledge or inherent non-determinism,

the information about real systems is based on approximations

- to model hybrid real-time and discrete-time interacting components,

these systems are frequently studied in interaction with discrete controllers, or with interactive environments having continuous behavior

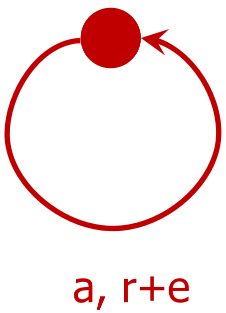
- to abstract complex continuous-time and continuous-space systems

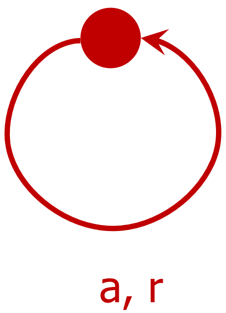
the real systems are reactive systems with continuous behaviour (in space and time)

Motivation

In this context, the stochastic/probabilistic bisimulation is a too strict concept

- the interest is to understand not whether two systems have identical behaviours, but when two systems have similar behaviours (up to an observational error)
- bisimulation => pseudometric that measures how similar two systems are from the point of view of their behaviours
- Model checking => property evaluation: instead of deciding whether "P⊨f", one measures "P⊨f" giving an observational error (granularity).





Overview

- We focus on continuous-time and continuous-space Markov processes (CMPs)
- We introduce the Continuous Markovian Logic (CML), a multimodal logic that characterizes the stochastic bisimulation. We provide complete Hilbert-style axiomatizations for CMLs and prove the finite model property
- We define an approximation of the satisfiability relation that induces:
  - a bisimulation pseudodistance on CMPs
  - a syntactic pseudodistance on logical formulas
- The pseudodistances are used to state the Strong Robustness Theorem and the finite model construction to approximate it in the form of the Weak Robustness Theorem

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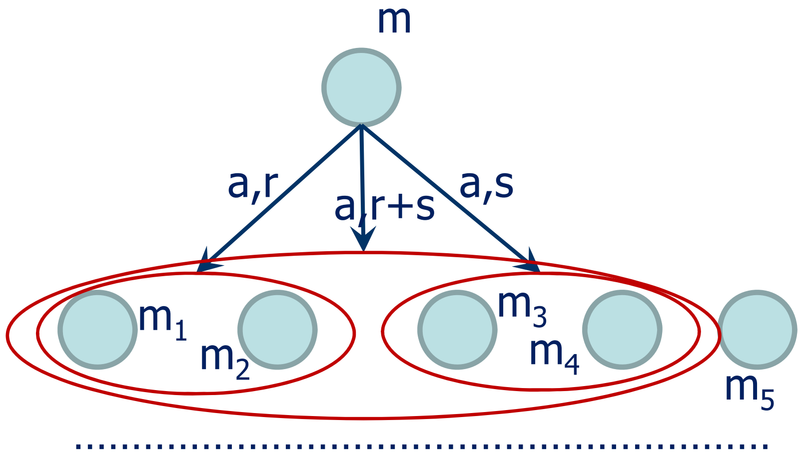
- The complete axiomatization allows the transfer of topological properties between the space of CMPs and the space of logical formulas.

Labelled Markov kernel

A tuple $\mathcal{M}=(M,\Sigma,A,\{R_a | a\in A\})$ where
- (M,Σ) is an analytic set (measurable space)
- Σ is the Borel-algebra generated by the topology
- A is a set of labels
- for each $a\in A$, $R_a : M\times\Sigma \to [0,1]$ is such that

  $R_a(m,-)$ - (sub-)probability measure on (M,Σ)

  $R_a(-,S)$ - measurable function



(P. Panangaden, *Labelled Markov Processes*, 2009.)

Equivalent definition:

A tuple $\mathcal{M}=(M,\Sigma,\theta)$ where $\theta\in[\![M \to \Pi(M,\Sigma)]\!]^A$

$\theta_a : M \to \Pi(M,\Sigma)$, $\theta_a(m)\in\Pi(M,\Sigma)$, $\theta_a(m)(S)\in [0,1]$

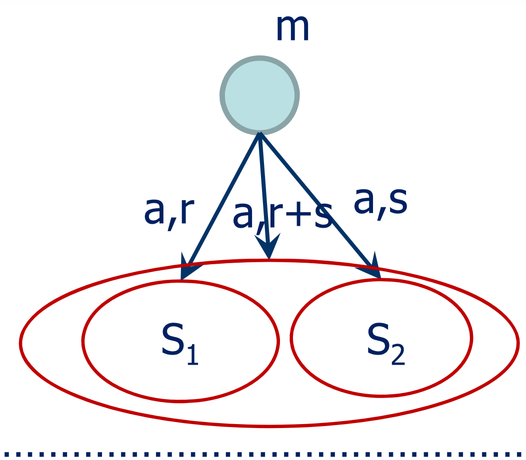
$\Pi(M,\Sigma)$ is a measurable space with the sigma-algebra generated, for arbitrary $S\in\Sigma$ and $r\in\mathbb{Q}$, by

$$\{\mu\in\Pi(M,\Sigma) \mid \mu(S)\leq r\}.$$

(E. Doberkat, Stochastic Relations, 2007.)

Continuous (Labelled) Markov kernel

A tuple $\mathcal{M}=(M,\Sigma,A,\{R_a|a\in A\})$ where
- (M,Σ) is an analytic set (measurable space)
- A is a set of labels
- for each $a\in A$, $R_a:M\times\Sigma \to[0,\infty)$ is such that
  - $R_a(m,-)$ – a measure on (M,Σ)
  - $R_a(-,S)$ – a measurable function



- $R_a(m,S)=r \in[0,+\infty)$ - the rate of an exponentially distributed random variable that characterizes the time of a-transitions from m to arbitrary elements of S.
- the probability of the *transition within time t* is given by the cumulative distribution function

$$P(t)= 1- e^{-rt}$$

Equivalent definition:

A tuple $\mathcal{M}=(M,\Sigma,\theta)$, where $\theta\in[\![M \to \Delta(M,\Sigma)]\!]^A$

$\theta_a: M \to \Delta(M,\Sigma)$, $\theta_a(m)\in\Delta(M,\Sigma)$, $\theta_a(m)(S)\in [0,+\infty)$

Continuous Markov process $(\mathcal{M},m)$, $m\in M$

Stochastic/Probabilistic Bisimulation

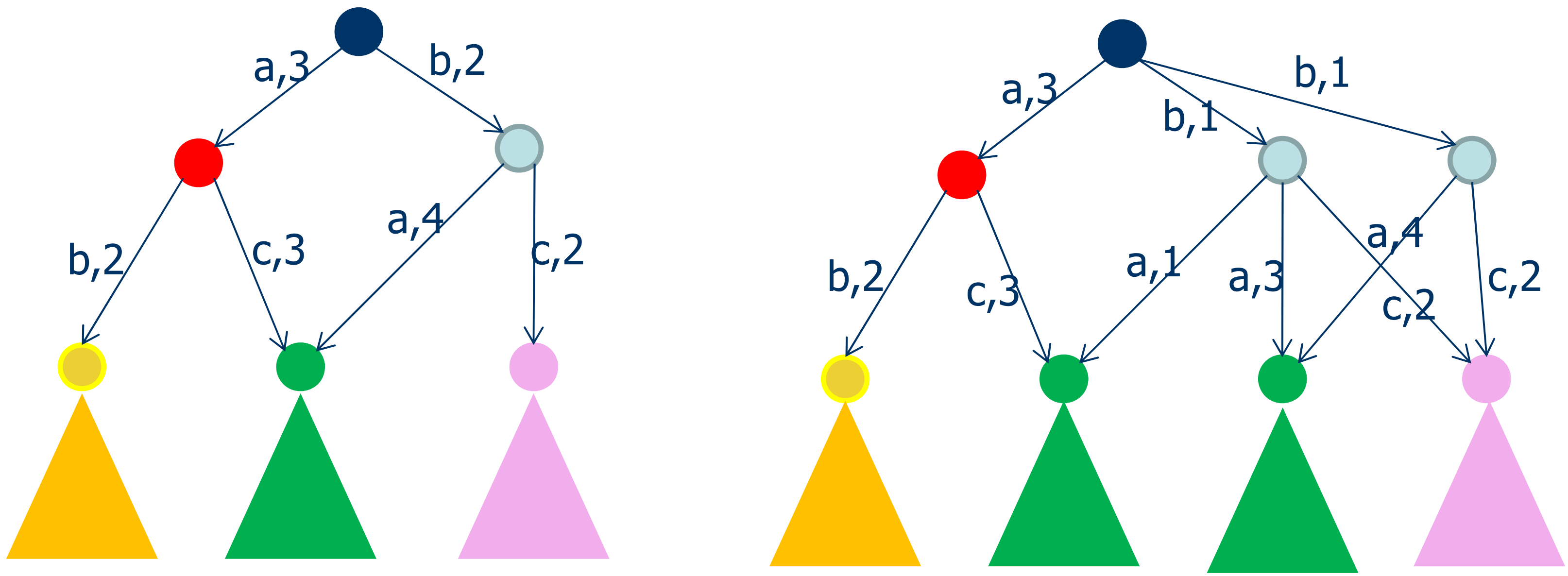
Given a **probabilistic/stochastic (Markovian) system** $\mathcal{M}=(M,\Sigma,\theta)$, a bisimulation relation is an equivalence relation $\sim \subseteq M\times M$ such that whenever $m_1 \sim m_2$, for arbitrary $S\in\Sigma(\sim)$ and $a\in A$

- If $m_1 \xrightarrow{a,p} S$, then $m_2 \xrightarrow{a,p} S$ and
- If $m_2 \xrightarrow{a,p} S$, then $m_1 \xrightarrow{a,p} S$.

$\theta_a(m)(S)=\theta_a(m')(S)$

K. G. Larsen and A. Skou. *Bisimulation through probabilistic testing*, I&C 1991

P. Panangaden , *Labelled Markov Processes*, 2009.



Continuous Markovian Logic

Syntax: CML(A)

$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f \qquad r \in \mathbb{Q}_+ \; a \in A$

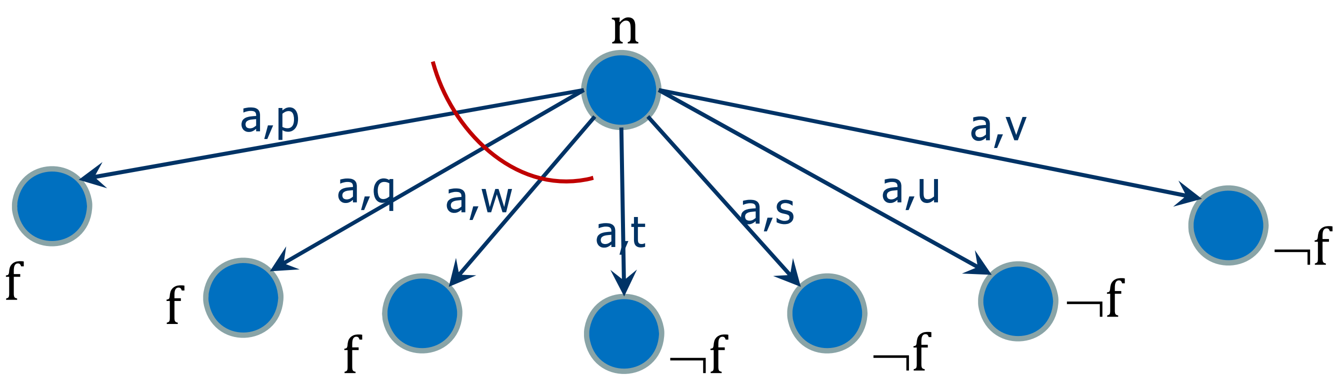
Semantics: Let $(m, \mathcal{M})$ be an arbitrary CMP with $\mathcal{M} = (M, \Sigma, \theta)$.

$(m,\mathcal{M}) \models T$ always

$(m,\mathcal{M}) \models \neg f$ iff $(m,\mathcal{M}) \not\models f$

$(m,\mathcal{M}) \models f_1 \wedge f_2$ iff $(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$

$(m,\mathcal{M}) \models L^a_r f$ iff $\theta_a(m)([f]) \geq r$, where $[f] = \{n \in M \mid (n,\mathcal{M}) \models f\}$



Continuous Markovian Logic

Syntax: $CML^+(A)$

$$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f \mid M^a_r f \qquad r \in \mathbb{Q}_+ \; a \in A$$

Semantics: Let $(m,\mathcal{M})$ be an arbitrary CMP with $\mathcal{M}=(M,\Sigma,\theta)$.

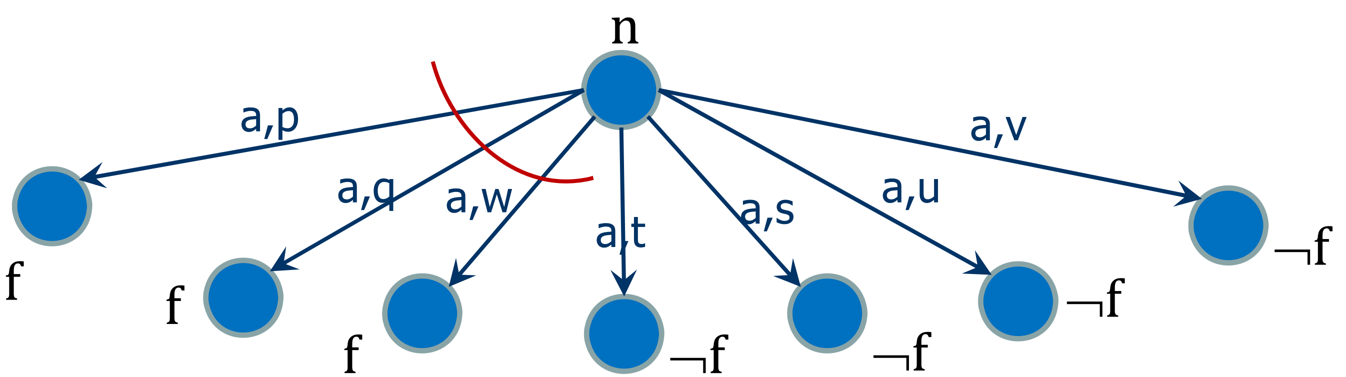
$(m,\mathcal{M}) \models T$ always

$(m,\mathcal{M}) \models \neg f$ iff $(m,\mathcal{M}) \not\models f$

$(m,\mathcal{M}) \models f_1 \wedge f_2$ iff $(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$

$(m,\mathcal{M}) \models L^a_r f$ iff $\theta_a(m)([f]) \geq r$

$(m,\mathcal{M}) \models M^a_r f$ iff $\theta_a(m)([f]) \leq r$, where $[f]=\{n \in M \mid (n,\mathcal{M}) \models f\}$



Continuous Markovian Logic

Syntax: CML(A) & CML$^+$(A)

$$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f \mid M^a_r f \qquad r \in \mathbb{Q}_+ \; a \in A$$

Semantics: Let $(m,\mathcal{M})$ be an arbitrary CMP with $\mathcal{M}=(M,\Sigma,\theta)$.

$(m,\mathcal{M}) \models T$	always
$(m,\mathcal{M}) \models \neg f$	iff $(m,\mathcal{M}) \not\models f$
$(m,\mathcal{M}) \models f_1 \wedge f_2$	iff $(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$
$(m,\mathcal{M}) \models L^a_r f$	iff $\theta_a(m)([f]) \geq r$
$(m,\mathcal{M}) \models M^a_r f$	iff $\theta_a(m)([f]) \leq r$, where $[f]=\{n \in M \mid (n,\mathcal{M}) \models f\}$

Theorem: For arbitrary continuous Markov processes $(m,\mathcal{M})$ and $(n,\mathcal{H})$, the following assertions are equivalent

(i) $(m,\mathcal{M}) \sim (n,\mathcal{H})$,

(ii) $\forall f \in CML(A)$, $(m,\mathcal{M}) \models f$ iff $(n,\mathcal{H}) \models f$,

(iii) $\forall f \in CML^+(A)$, $(m,\mathcal{M}) \models f$ iff $(n,\mathcal{H}) \models f$.

(P. Panangaden, *Labelled Markov Processes*, 2009.)

Modal Probabilistic Logic versus Continuous Markovian Logic

$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f \mid M^a_r f \qquad a \in A$

MPL(A) for LMPs

$\mathcal{M}=(M,\Sigma,\theta),\ \theta \in [\![M \to \Pi(M,\Sigma)]\!]^A$
$S \in \Sigma,\ \theta_a(m)(S) \in [0,1]$

$\vdash M^a_r f \leftrightarrow L^a_{1-r} \neg f$

$\vdash L^a_r f \leftrightarrow \neg L^a_s \neg f, \quad r+s>1$

$\vdash$ [*If* a *is active*] $\to L^a_r T$
$\vdash L^a_s f \to L^a_r T$

For a fixed $q \in \mathbb{N}$ the set $\{p/q \in [0,1] \mid p \in \mathbb{N}\}$ is finite

CML(A) for CMPs

$\mathcal{M}=(M,\Sigma,\theta),\ \theta \in [\![M \to \Delta(M,\Sigma)]\!]^A$
$S \in \Sigma,\ \theta_a(m)(S) \in [0,+\infty)$

$M^a_r f$ and $L^a_s f$ are independent operators

$\vdash L^a_{s+r} f \to \neg M^a_r f,\ \ s>0$
$\vdash M^a_{s+r} f \to \neg L^a_r f,\ \ s>0$

$\vdash \neg L^a_r f \to M^a_r f$
$\vdash \neg M^a_r f \to L^a_r f$

For a fixed $q \in \mathbb{N}$ the set $\{p/q \in [0,+\infty) \mid p \in \mathbb{N}\}$ is not finite

K.G. Larsen, A. Skou. *Bisimulation through probabilistic testing*, 1991.
R. Fagin, J.Y. Halpern, Reasoning about Knowledge and Probability, 1994
A. Heifetz, P. Mongin, Probability Logic for Type Spaces, 2001
C. Zhou, *A complete deductive system for probability logic with application to Harsanyi type spaces*, 2007.

Axiomatic Systems

CML(A)

(A1) $\vdash L^a_0 f$

(A2) $\vdash L^a_{r+s} f \rightarrow L^a_r f$

(A3) $\vdash L^a_r(f \wedge g) \wedge L^a_s(f \wedge \neg g) \rightarrow L^a_{r+s} f$

(A4) $\vdash \neg L^a_r(f \wedge g) \wedge \neg L^a_s(f \wedge \neg g) \rightarrow \neg L^a_{r+s} f$

CML⁺(A)

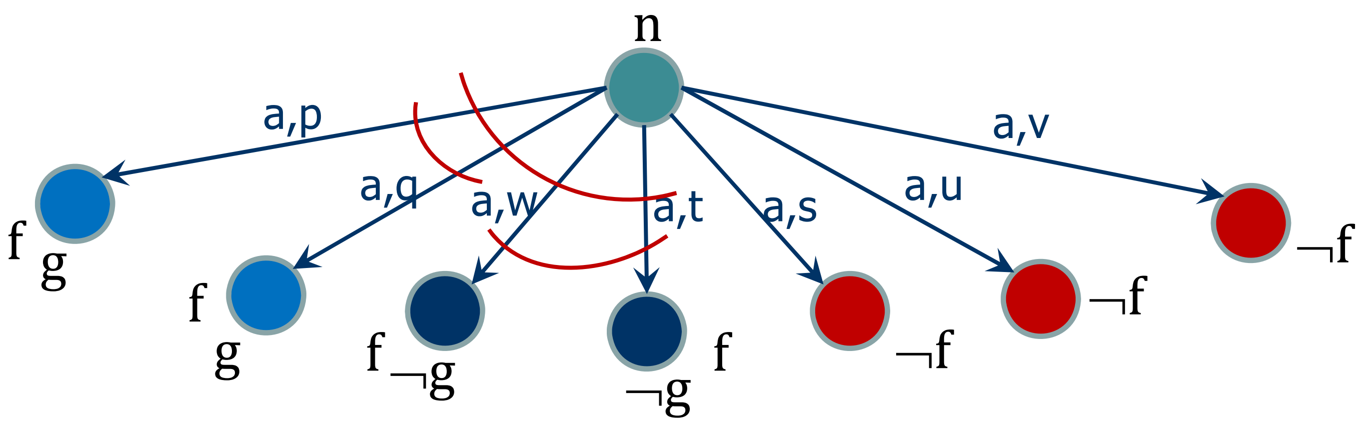
(B1) $\vdash L^a_0 f$

(B2) $\vdash L^a_{r+s} f \rightarrow \neg M^a_r f$, $s>0$

(B3) $\vdash \neg L^a_r f \rightarrow M^a_r f$

(B4) $\vdash \neg L^a_r(f \wedge g) \wedge \neg L^a_s(f \wedge \neg g) \rightarrow \neg L^a_{r+s} f$

(B5) $\vdash \neg M^a_r(f \wedge g) \wedge \neg M^a_s(f \wedge \neg g) \rightarrow \neg M^a_{r+s} f$



Axiomatic Systems

CML(A)

(A1) $\vdash L^a_0 f$

(A2) $\vdash L^a_{r+s} f \rightarrow L^a_r f$

(A3) $\vdash L^a_r(f \wedge g) \wedge L^a_s(f \wedge \neg g) \rightarrow L^a_{r+s} f$

(A4) $\vdash \neg L^a_r(f \wedge g) \wedge \neg L^a_s(f \wedge \neg g) \rightarrow \neg L^a_{r+s} f$

(R1) If $\vdash f \rightarrow g$, then $\vdash L^a_r f \rightarrow L^a_r g$

(R2) If $\forall r<s$, $\vdash f \rightarrow L^a_r g$, then $\vdash f \rightarrow L^a_s g$

(R3) If $\forall r>s$, $\vdash f \rightarrow L^a_r g$, then $\vdash f \rightarrow \neg T$

CML$^+$(A)

(B1) $\vdash L^a_0 f$

(B2) $\vdash L^a_{r+s} f \rightarrow \neg M^a_r f$, $s>0$

(B3) $\vdash \neg L^a_r f \rightarrow M^a_r f$

(B4) $\vdash \neg L^a_r(f \wedge g) \wedge \neg L^a_s(f \wedge \neg g) \rightarrow \neg L^a_{r+s} f$

(B5) $\vdash \neg M^a_r(f \wedge g) \wedge \neg M^a_s(f \wedge \neg g) \rightarrow \neg M^a_{r+s} f$

(S1) If $\vdash f \rightarrow g$, then $\vdash L^a_r f \rightarrow L^a_r g$

(S2) If $\forall r<s$, $\vdash f \rightarrow L^a_r g$, then $\vdash f \rightarrow L^a_s g$

(S3) If $\forall r>s$, $\vdash f \rightarrow M^a_r g$, then $\vdash f \rightarrow M^a_s g$

(S4) If $\forall r>s$, $\vdash f \rightarrow L^a_r g$, then $\vdash f \rightarrow \neg T$

A. Heifetz, P. Mongin, Probability Logic for Type Spaces, 2001
C. Kupke, D. Pattinson. On Modal Logics of Linear Inequalities, AiML 2010.

Metaproperties

Metatheorem [Small model property]:

If f is consistent (in CML(A) or $CML^+(A)$), there exists a CMP $(m,\mathcal{M}^e_f)$ that satisfies f.

The support of $\mathcal{M}^e_f$ is finite of cardinality bound by the dimension of f; the construction of $\mathcal{M}^e_f$ is parametric ($e>0$) and depends on the *granularity* of f.

The granularity of a set $S\subseteq\mathbb{Q}^+$ is the least common denominator of the elements of S.

Metatheorem [Soundness & Weak Completeness]:

The axiomatic system of CML(A) and $CML^+(A)$ are sound and complete w.r.t. the Markovian semantics,

$$\vdash f \text{ iff } \models f.$$

Similar Behaviours

- Stochastic bisimulation equates CMPs with identical stochastic behaviours
- CMLs are multimodal logics that characterize stochastic bisimulation
- CMLs are completely axiomatized for CMP-semantics
- We have a clear intuition of what a distance between CMPs should be

Similar Behaviours

Classical Logic	Generalization
Truth values $\{0,1\}$	Interval $[0,1]$
Propositional function	Measurable function
State	Measure
The satisfiability relation $\models$	Integration $\int$

D. Kozen, A Probabilistic PDL, 1985.

Similar Behaviours

The satisfiability relation is replaced by a pseudometric over the space of CMPs.

$d:\wp\times\mathcal{L} \to [0,1]$ ← $\models:\wp\times\mathcal{L} \to \{0,1\}$

$d((m,\mathcal{M}),T)=0$

$d((m,\mathcal{M}),\neg f)=1- d((m,\mathcal{M}),f)$

$d((m,\mathcal{M}),f_1\wedge f_2)=\max\{d((m,\mathcal{M}),f_1),d((m,\mathcal{M}),f_2)\}$

$d((m,\mathcal{M}), L^a_r f)=<r, \theta_a(m)([f])>$

$d((m,\mathcal{M}), M^a_r f)=<\theta_a(m)([f]), r>$

$(m,\mathcal{M})\models T$ always

$(m,\mathcal{M})\models\neg f$ iff $(m,\mathcal{M})\not\models f$

$(m,\mathcal{M})\models f_1\wedge f_2$ iff $(m,\mathcal{M})\models f_1$, $(m,\mathcal{M})\models f_2$

$(m,\mathcal{M})\models L^a_r f$ iff $\theta_a(m)([f])\geq r$

$(m,\mathcal{M})\models M^a_r f$ iff $\theta_a(m)([f])\leq r$,

$$<r,s>= \begin{cases} (r-s)/r, & \text{if } r>s \\ 0, & \text{otherwise} \end{cases}$$

<u>Example:</u>

$(m,\mathcal{M})\models L^a_r f \Rightarrow \theta_a(m)([f])\geq r \Rightarrow d((m,\mathcal{M}), L^a_r f)=0$

$(m,\mathcal{M})\not\models L^a_r f \Rightarrow \theta_a(m)([f])<r \Rightarrow d((m,\mathcal{M}), L^a_r f)>0$

Similar Behaviours

$d:\wp \times \mathcal{L} \rightarrow [0,1]$

$d((m,\mathcal{M}),T)=0$

$d((m,\mathcal{M}),\neg f)=1- d((m,\mathcal{M}),f)$

$d((m,\mathcal{M}),f_1 \wedge f_2)=\max\{d((m,\mathcal{M}),f_1),d((m,\mathcal{M}),f_2)\}$

$d((m,\mathcal{M}), L^a_r f)=\langle r, \theta_a(m)([f])\rangle$

$d((m,\mathcal{M}), M^a_r f)=\langle \theta_a(m)([f]), r\rangle$

$$\langle r,s\rangle = \begin{cases} (r-s)/r\ , & \text{if } r>s \\ 0, & \text{otherwise} \end{cases}$$

$D:\wp \times \wp \rightarrow [0,1]$,

$$D((m,\mathcal{M}), (m',\mathcal{M}')) = \sup\{|d((m,\mathcal{M}),f) - d((m',\mathcal{M}'),f)|,\ f \in \mathcal{L}\}$$

$\delta:\mathcal{L} \times \mathcal{L} \rightarrow [0,1]$,

$$\delta(f,f') = \sup\{|d((m,\mathcal{M}),f) - d((m,\mathcal{M}),f')|,\ (m,\mathcal{M}) \in \wp\}$$

Metaproperties

Theorem [Strong Robustness]:

For arbitrary $f, f' \in \mathcal{L}$, and arbitrary $(m, \mathcal{M}) \in \wp$,

$$d((m,\mathcal{M}),f') \leq d((m,\mathcal{M}),f) + \delta(f,f')$$

$\delta^*: \mathcal{L} \times \mathcal{L} \rightarrow [0,1]$,

$$\delta^*(f,f') = \sup\{|d((m,\mathcal{M}^e_{f\wedge f'}),f) - d((m,\mathcal{M}_{f\wedge f'}),f')|, m \in \sup(\mathcal{M}_{f\wedge f'})\}$$

where $\mathcal{M}^e_{f\wedge f'}$ is the finite model of $\sim(f\wedge f')$ of parameter $e>0$.

Lemma: For arbitrary $f, f' \in \mathcal{L}$

$$\delta(f,f') \leq \delta^*(f,f') + 2/e$$

Theorem [Weak Robustness]:

For arbitrary $f, f' \in \mathcal{L}$, and arbitrary $(m, \mathcal{M}) \in \wp$,

$$d((m,\mathcal{M}),f') \leq d((m,\mathcal{M}),f) + \delta^*(f,f') + 2/e$$

Towards a metric semantics

Working hypothesis:

- Let $(\wp, D)$ be a pseudometrizable space of Markovian systems such that D converges to bisimulation;
- Let $\mathcal{L}$ be the continuous Markovian logic (that characterizes the bisimulation and is completely axiomatized for $\wp$)

$$\mathcal{L} \quad f := T \mid \neg f \mid f \wedge f \mid L^a_r f \mid M^a_r f$$

$$\mathcal{L}(+) \quad g := T \mid g \wedge g \mid L^a_r f \mid M^a_r f$$

$$\mathcal{L}(-) = \mathcal{L} - \mathcal{L}(+)$$

<u>Theorem:</u> If $\vdash f \leftrightarrow g$, then $\delta(f,g)=0$.

<u>Theorem:</u> If $\delta(f,g)=0$ and $f \in \mathcal{L}(+)$, then $\vdash g \rightarrow f$.

<u>Theorem:</u> If $\delta(f,g)=0$ and $f,g \in \mathcal{L}(+)$, then $\vdash f \leftrightarrow g$.

In this context, δ is a pseudometric that measure the syntactical equivalence on $\mathcal{L}(+)$.

Future work: some dualities

Working hypothesis:

- Let $(\wp, D)$ be a pseudometrizable space of Markovian systems such that D converges to bisimulation;
- Let $\mathcal{L}$ be the continuous Markovian logic (that characterizes the bisimulation and is completely axiomatized for $\wp$)
- $\mathcal{L}$ has a *canonical model* $\tilde{\Omega}=(\Omega,2^{\Omega},\theta)$, where each $F\in\Omega$ is a maximally consistent set of formulas: for each CMP $(\mathcal{M},m)$ there exists a unique $F\in\Omega$ such that
  $$(m,\mathcal{M})\sim(F,\tilde{\Omega}).$$

In fact, $F=\{f\in\mathscr{L}, (m,\mathcal{M})\models f\}$.

If for an arbitrary distance D we use D_H to denote the Hausdorff distance associated to D, then the complete axiomatization suggest the following conjectures.

Conjecture1:	$(D_H)_H=D$

Conjecture2:	$(\delta_H)_H=\delta$