Continuous Markovian Logic -
From Complete Axiomatization to the Metric Space of Formulas

Luca Cardelli
Microsoft Research Cambridge, UK

Kim G. Larsen
Aalborg University, Denmark

Radu Mardare
Aalborg University, Denmark
Motivation

Complex systems are often modelled as stochastic processes

biological and ecological systems, physical systems, social systems, financial systems

• to encapsulate a lack of knowledge or inherent non-determinism,
the information about real systems is based on approximations

• to model hybrid real-time and discrete-time interacting components,
these systems are frequently studied in interaction with discrete controllers, or with
interactive environments having continuous behavior

• to abstract complex continuous-time and continuous-space systems
the real systems are reactive systems with continuous behaviour (in space and time)
Motivation

In this context, the stochastic/probabilistic bisimulation is a too strict concept

- the interest is to understand not whether two systems have identical behaviours, but when two systems have similar behaviours (up to an observational error)

- bisimulation => pseudometric that measures how similar two systems are from the point of view of their behaviours

- Model checking => property evaluation: instead of deciding whether “$P \models f$”, one measures “$P \models f$” giving an observational error (granularity).
Overview

- We focus on continuous-time and continuous-space Markov processes (CMPs)

- We introduce the Continuous Markovian Logic (CML), a multimodal logic that characterizes the stochastic bisimulation. We provide complete Hilbert-style axiomatizations for CMLs and prove the finite model property.

- We define an approximation of the satisfiability relation that induces:
  - a bisimulation pseudodistance on CMPs
  - a syntactic pseudodistance on logical formulas

- The pseudodistances are used to state the Strong Robustness Theorem and the finite model construction to approximate it in the form of the Weak Robustness Theorem.

- The complete axiomatization allows the transfer of topological properties between the space of CMPs and the space of logical formulas.
Labelled Markov kernel

A tuple $\mathcal{M} = (M, \Sigma, A, \{R_a | a \in A\})$ where
- $(M, \Sigma)$ is an analytic set (measurable space)
- $\Sigma$ is the Borel-algebra generated by the topology
- $A$ is a set of labels
- for each $a \in A$, $R_a : M \times \Sigma \rightarrow [0,1]$ is such that
  $R_a(m,-)$ - (sub-)probability measure on $(M, \Sigma)$
  $R_a(-,S)$ - measurable function


Equivalent definition:

A tuple $\mathcal{M} = (M, \Sigma, \theta)$ where $\theta \in \Pi(M \rightarrow \Pi(M, \Sigma))^A$

$\theta_a : M \rightarrow \Pi(M, \Sigma), \quad \theta_a(m) \in \Pi(M, \Sigma), \quad \theta_a(m)(S) \in [0,1]$

$\Pi(M, \Sigma)$ is a measurable space with the sigma-algebra generated, for arbitrary $S \in \Sigma$ and $r \in \mathbb{Q}$, by

$\{\mu \in \Pi(M, \Sigma) | \mu(S) \leq r\}$.

(E. Doberkat, *Stochastic Relations*, 2007.)
Continuous (Labelled) Markov kernel

A tuple \( \mathcal{M} = (M, \Sigma, A, \{ R_a | a \in A \}) \) where
- \( (M, \Sigma) \) is an analytic set (measurable space)
- \( A \) is a set of labels
- for each \( a \in A \), \( R_a : M \times \Sigma \to [0, \infty) \) is such that
  \( R_a(m, -) \) – a measure on \( (M, \Sigma) \)
  \( R_a(\cdot, S) \) – a measurable function

- \( R_a(m, S) = r \in [0, +\infty) \) - the rate of an exponentially distributed random variable that characterizes the time of \( a \)-transitions from \( m \) to arbitrary elements of \( S \).
- the probability of the transition within time \( t \) is given by the cumulative distribution function \( P(t) = 1 - e^{-rt} \)

Equivalent definition:

A tuple \( \mathcal{M} = (M, \Sigma, \theta) \), where \( \theta \in [M \to \Delta(M, \Sigma)]^A \)

\( \theta_a : M \to \Delta(M, \Sigma), \theta_a(m) \in \Delta(M, \Sigma), \theta_a(m)(S) \in [0, +\infty) \)

Continuous Markov process \( (\mathcal{M}, m), m \in M \)
Stochastic/Probabilistic Bisimulation

Given a **probabilistic/stochastic (Markovian) system** $\mathcal{M} = (M, \Sigma, \theta)$, a bisimulation relation is an equivalence relation $\sim \subseteq M \times M$ such that whenever $m_1 \sim m_2$, for arbitrary $S \in \Sigma(\sim)$ and $a \in A$

- If $m_1 \xrightarrow{a, p} S$, then $m_2 \xrightarrow{a, p} S$ and $\theta_a(m)(S) = \theta_a(m')(S)$
- If $m_2 \xrightarrow{a, p} S$, then $m_1 \xrightarrow{a, p} S$.


Continuous Markovian Logic

**Syntax:** \( \text{CML}(A) \)

\[
f := T \mid \neg f \mid f_1 \land f_2 \mid L^a_r f \quad r \in \mathbb{Q}_+ \quad a \in A
\]

**Semantics:** Let \((m, \mathcal{M})\) be an arbitrary CMP with \(\mathcal{M}=(M, \Sigma, \theta)\).

\[
\begin{align*}
(m, \mathcal{M}) \models T & \quad \text{always} \\
(m, \mathcal{M}) \models \neg f & \quad \text{iff } (m, \mathcal{M}) \not\models f \\
(m, \mathcal{M}) \models f_1 \land f_2 & \quad \text{iff } (m, \mathcal{M}) \models f_1 \text{ and } (m, \mathcal{M}) \models f_2 \\
(m, \mathcal{M}) \models L^a_r f & \quad \text{iff } \theta_a(m)([f]) \geq r, \text{ where } [f] = \{n \in M \mid (n, \mathcal{M}) \models f\}
\end{align*}
\]
Continuous Markovian Logic

Syntax: CML\(^+(A)\)

\[ f := T \mid \neg f \mid f_1 \land f_2 \mid L^a_r f \mid M^a_r f \quad r \in \mathbb{Q}_+ \quad a \in A \]

Semantics: Let \((m,M)\) be an arbitrary CMP with \(M=(M,\Sigma,\theta)\).

\[(m,M) \models T \quad \text{always} \]
\[(m,M) \models \neg f \quad \text{iff} \quad (m,M) \not\models f \]
\[(m,M) \models f_1 \land f_2 \quad \text{iff} \quad (m,M) \models f_1 \quad \text{and} \quad (m,M) \models f_2 \]
\[(m,M) \models L^a_r f \quad \text{iff} \quad \theta_a(m)(\{f\}) \geq r \]
\[(m,M) \models M^a_r f \quad \text{iff} \quad \theta_a(m)(\{f\}) \leq r, \quad \text{where} \quad [f] = \{n \in M \mid (n,M) \models f\} \]
Continuous Markovian Logic

Syntax: $\text{CML(A)} & \text{CML}^+(A)$

$$f := T | \neg f | f_1 \land f_2 | L^a f | M^a f \quad r \in \mathbb{Q}_+ \ a \in A$$

Semantics: Let $(m, M)$ be an arbitrary CMP with $M = (M, \Sigma, \theta)$.

$$(m, M) \models T \quad \text{always}$$
$$(m, M) \models \neg f \quad \text{iff} \quad (m, M) \not\models f$$
$$(m, M) \models f_1 \land f_2 \quad \text{iff} \quad (m, M) \models f_1 \quad \text{and} \quad (m, M) \models f_2$$
$$(m, M) \models L^a f \quad \text{iff} \quad \theta_a(m)([f]) \geq r$$
$$(m, M) \models M^a f \quad \text{iff} \quad \theta_a(m)([f]) \leq r, \text{ where } [f] = \{ n \in M \mid (n, M) \models f \}$$

Theorem: For arbitrary continuous Markov processes $(m, M)$ and $(n, H)$, the following assertions are equivalent

(i) $(m, M) \sim (n, H)$,

(ii) $\forall f \in \text{CML(A)}, (m, M) \models f \quad \text{iff} \quad (n, H) \models f$,

(iii) $\forall f \in \text{CML}^+(A), (m, M) \models f \quad \text{iff} \quad (n, H) \models f$.

# Modal Probabilistic Logic versus Continuous Markovian Logic

$$f := T \mid \neg f \mid f_1 \land f_2 \mid L^a_r f \mid M^a_r f \quad \forall a \in A$$

## MPL(A) for LMPs

$\mathcal{M} = (M, \Sigma, \theta)$, $\theta \in [M \rightarrow \Pi(M, \Sigma)]^A$

$S \in \Sigma$, $\theta_a(m)(S) \in [0,1]$

\[\vdash M^a_r f \leftrightarrow L^a_{1-r} \neg f\]

\[\vdash L^a_r f \leftrightarrow \neg L^a_{s} \neg f, \quad r+s>1\]

\[\vdash [If a is active] \rightarrow L^a_r T\]

\[\vdash L^a_s f \rightarrow L^a_r T\]

**For a fixed $q \in \mathbb{N}$ the set**

\[\{p/q \in [0,1] \mid p \in \mathbb{N}\}\] **is finite**

## CML(A) for CMPs

$\mathcal{M} = (M, \Sigma, \theta)$, $\theta \in [M \rightarrow \Delta(M, \Sigma)]^A$

$S \in \Sigma$, $\theta_a(m)(S) \in [0,+,\infty)$

$M^a_r f$ and $L^a_s f$ are independent operators

\[\vdash L^a_{s+r} f \rightarrow \neg M^a_r f, \quad s>0\]

\[\vdash M^a_{s+r} f \rightarrow \neg L^a_r f, \quad s>0\]

\[\vdash \neg L^a_r f \rightarrow M^a_r f\]

\[\vdash \neg M^a_r f \rightarrow L^a_r f\]

**For a fixed $q \in \mathbb{N}$ the set**

\[\{p/q \in [0,+,\infty) \mid p \in \mathbb{N}\}\] **is not finite**

---


Axiomatic Systems

CML(A)

(A1) ⊢ La₀f
(A2) ⊢ Laᵣ₊sf → Laᵣf
(A3) ⊢ Laᵣ(f ∧ g) ∧ Laₛ(f ∧ ¬g) → Laᵣ₊sf
(A4) ⊢ ¬Laᵣ(f ∧ g) ∧ ¬Laₛ(f ∧ ¬g) → ¬Laᵣ₊sf

CML⁺(A)

(B1) ⊢ La₀f
(B2) ⊢ Laᵣ₊sf → ¬Maᵣf , s>0
(B3) ⊢ ¬Laᵣf → Maᵣf
(B4) ⊢ ¬Laᵣ(f ∧ g) ∧ ¬Laₛ(f ∧ ¬g) → ¬Laᵣ₊sf
(B5) ⊢ ¬Maᵣ(f ∧ g) ∧ ¬Maₛ(f ∧ ¬g) → ¬Maᵣ₊sf
Axiomatic Systems

**CML(A)**

(A1) ⊢ L_{0}^{a}f
(A2) ⊢ L_{r+s}^{a}f → L_{r}^{a}f
(A3) ⊢ L_{r}^{a}(f \land g) \land L_{s}^{a}(f \land \neg g) → L_{r+s}^{a}f
(A4) ⊢ \neg L_{r}^{a}(f \land g) \land \neg L_{s}^{a}(f \land \neg g) → \neg L_{r+s}^{a}f

(R1) If ⊢ f → g, then ⊢ L_{r}^{a}f → L_{r}^{a}g
(R2) If ∀ r<s, ⊢ f → L_{s}^{a}g, then ⊢ f → L_{s}^{a}g
(R3) If ∀ r>s, ⊢ f → L_{s}^{a}g, then ⊢ f → \neg T

**CML+(A)**

(B1) ⊢ L_{0}^{a}f
(B2) ⊢ L_{r+s}^{a}f → \neg M_{r}^{a}f, s>0
(B3) ⊢ \neg L_{r}^{a}f → M_{r}^{a}f
(B4) ⊢ \neg L_{r}^{a}(f \land g) \land \neg L_{s}^{a}(f \land \neg g) → \neg L_{r+s}^{a}f
(B5) ⊢ \neg M_{r}^{a}(f \land g) \land \neg M_{s}^{a}(f \land \neg g) → \neg M_{r+s}^{a}f

(S1) If ⊢ f → g, then ⊢ L_{r}^{a}f → L_{r}^{a}g
(S2) If ∀ r<s, ⊢ f → L_{s}^{a}g, then ⊢ f → L_{s}^{a}g
(S3) If ∀ r>s, ⊢ f → M_{r}^{a}g, then ⊢ f → M_{s}^{a}g
(S4) If ∀ r>s, ⊢ f → L_{s}^{a}g, then ⊢ f → \neg T

A. Heifetz, P. Mongin, Probability Logic for Type Spaces, 2001
Metaproperties

**Metatheorem [Small model property]:**
If \( f \) is consistent (in CML(A) or CML\(^+(A)\)), there exists a CMP \((m, M^e_f)\) that satisfies \( f \). The support of \( M^e_f \) is finite of cardinality bound by the dimension of \( f \); the construction of \( M^e_f \) is parametric \((e>0)\) and depends on the **granularity** of \( f \).

The granularity of a set \( S \subseteq \mathbb{Q}^+ \) is the least common denominator of the elements of \( S \).

**Metatheorem [Soundness & Weak Completeness]:**
The axiomatic system of CML(A) and CML\(^+(A)\) are sound and complete w.r.t. the Markovian semantics,

\[ \models f \iff \not \models f. \]
Similar Behaviours

- Stochastic bisimulation equates CMPs with identical stochastic behaviours
- CMLs are multimodal logics that characterize stochastic bisimulation
- CMLs are completely axiomatized for CMP-semantics
- We have a clear intuition of what a distance between CMPs should be
### Similar Behaviours

<table>
<thead>
<tr>
<th>Classical Logic</th>
<th>Generalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truth values {0,1}</td>
<td>Interval ([0,1])</td>
</tr>
<tr>
<td>Propositional function</td>
<td>Measurable function</td>
</tr>
<tr>
<td>State</td>
<td>Measure</td>
</tr>
<tr>
<td>The satisfiability relation (\models)</td>
<td>Integration (\int)</td>
</tr>
</tbody>
</table>

Similar Behaviours

The satisfiability relation is replaced by a pseudometric over the space of CMPs.

\[ d: \mathcal{M} \times \mathcal{L} \to [0,1] \quad \iff \quad \models: \mathcal{M} \times \mathcal{L} \to \{0,1\} \]

\[
\begin{align*}
    d((m, \mathcal{M}), T) &= 0 \\
    d((m, \mathcal{M}), \neg f) &= 1 - d((m, \mathcal{M}), f) \\
    d((m, \mathcal{M}), f_1 \land f_2) &= \max\{d((m, \mathcal{M}), f_1), d((m, \mathcal{M}), f_2)\} \\
    d((m, \mathcal{M}), La^rf) &= <r, \theta_a(m)([f])> \\
    d((m, \mathcal{M}), Ma^rf) &= <\theta_a(m)([f]), r> \\
\end{align*}
\]

\[
<\mathcal{M}, \mathcal{L}, r, s> = \begin{cases} 
    (r-s)/r, & \text{if } r > s \\
    0, & \text{otherwise}
\end{cases}
\]

**Example:**

\[
\begin{align*}
    (m, \mathcal{M}) \models L^a_r f & \implies \theta_a(m)([f]) \geq r \implies d((m, \mathcal{M}), L^a_r f) = 0 \\
    (m, \mathcal{M}) \npreceq L^a_r f & \implies \theta_a(m)([f]) < r \implies d((m, \mathcal{M}), L^a_r f) > 0
\end{align*}
\]
Similar Behaviours

\( d : \mathcal{O} \times \mathcal{L} \to [0,1] \)

\[
d((m,M),T)=0
\]

\[
d((m,M),\neg f)=1-d((m,M),f)
\]

\[
d((m,M),f_1 \& f_2)=\max\{d((m,M),f_1),d((m,M),f_2)\}
\]

\[
d((m,M), L^a f)=<r, \theta_a(m)(f)>
\]

\[
d((m,M), M^a f)=<\theta_a(m)(f), r>
\]

\[
\mathcal{D} : \mathcal{O} \times \mathcal{O} \to [0,1],
\]

\[
\mathcal{D}((m,M), (m',M')) = \sup\{|d((m,M),f) - d((m',M'),f)|, f \in \mathcal{L}\}
\]

\[
\delta : \mathcal{L} \times \mathcal{L} \to [0,1],
\]

\[
\delta(f,f') = \sup\{|d((m,M),f) - d((m,M),f')|, (m,M) \in \mathcal{O}\}
\]

\[
<r,s> = \begin{cases} 
(r-s)/r, & \text{if } r>s \\
0, & \text{otherwise}
\end{cases}
\]
**Metaproperties**

**Theorem [Strong Robustness]:**

For arbitrary $f, f' \in \mathcal{L}$, and arbitrary $(m, \mathcal{M}) \in \mathcal{O}$,

$$d((m, \mathcal{M}), f') \leq d((m, \mathcal{M}), f) + \delta(f, f')$$

**Theorem [Weak Robustness]:**

For arbitrary $f, f' \in \mathcal{L}$, and arbitrary $(m, \mathcal{M}) \in \mathcal{O}$,

$$d((m, \mathcal{M}), f') \leq d((m, \mathcal{M}), f) + \delta^*(f, f') + \frac{2}{e}$$

**Lemma:** For arbitrary $f, f' \in \mathcal{L}$

$$\delta(f, f') \leq \delta^*(f, f') + \frac{2}{e}$$

**Lemma:**

$$\delta^*: \mathcal{L} \times \mathcal{L} \to [0,1],$$

$$\delta^*(f, f') = \sup\{|d((m, \mathcal{M}_{e, f\wedge f'}^e), f) - d((m, \mathcal{M}_{f\wedge f'}^e), f')|, m \in \sup(\mathcal{M}_{f\wedge f'})\}$$

where $\mathcal{M}_{e, f\wedge f'}^e$ is the finite model of $\sim(f \wedge f')$ of parameter $e > 0$. 
Towards a metric semantics

Working hypothesis:

- Let $(\mathcal{P}, D)$ be a pseudometrizable space of Markovian systems such that $D$ converges to bisimulation;
- Let $\mathcal{L}$ be the continuous Markovian logic (that characterizes the bisimulation and is completely axiomatized for $\mathcal{P}$)

$$\mathcal{L} \gg f := T | \neg f | f \land f | L^a f | M^a f$$

$$\mathcal{L}(+) \gg g := T | g \land g | L^a f | M^a f$$

$$\mathcal{L}(-) = \mathcal{L} - \mathcal{L}(+)$$

**Theorem:** If $\vdash f \leftrightarrow g$, then $\delta(f,g)=0$.

**Theorem:** If $\delta(f,g)=0$ and $f \in \mathcal{L}(+)$, then $\vdash g \rightarrow f$.

**Theorem:** If $\delta(f,g)=0$ and $f, g \in \mathcal{L}(+)$, then $\vdash f \leftrightarrow g$.

In this context, $\delta$ is a pseudometric that measures the syntactical equivalence on $\mathcal{L}(+)$. 

Future work: some dualities

Working hypothesis:

• Let \((\mathcal{O},D)\) be a pseudometrizable space of Markovian systems such that \(D\) converges to bisimulation;

• Let \(L\) be the continuous Markovian logic (that characterizes the bisimulation and is completely axiomatized for \(\mathcal{O}\))

• \(L\) has a canonical model \(\Omega=(\Omega,2^\Omega,\theta)\), where each \(F\in\Omega\) is a maximally consistent set of formulas: for each CMP \((M,m)\) there exists a unique \(F\in\Omega\) such that \((m,M)\sim (F,\Omega)\).

In fact, \(F=\{f\in L, (m,M)\models f\}\).

If for an arbitrary distance \(D\) we use \(D_H\) to denote the Hausdorff distance associated to \(D\), then the complete axiomatization suggest the following conjectures.

\begin{itemize}
  \item **Conjecture1:** \((D_H)_H=D\)
  \item **Conjecture2:** \((\delta_H)_H=\delta\)
\end{itemize}