The Measurable Space of Stochastic Processes

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Motivation

Complex networks are often modelled as stochastic processes

- to encapsulate a lack of knowledge or inherent non-determinism,

E.g., natural systems (bio-chemical, ecological or physical)

- to model and analyse the global behaviours of systems containing components with both continuous-time and discrete-time evolutions,

E.g., embedded systems, communication networks, the Internet, service-oriented architectures, web-services, financial systems, market scenarios, etc.

Such systems are frequently modular in nature

- consist of modules which are systems in their own right,
- the modules interact, communicate and interrupt each other
- the global behaviour depends on the behaviours of the modules and on their links,
- the modules are easier to model, test, measure, analyse.

A possible approach: Stochastic Process Algebras

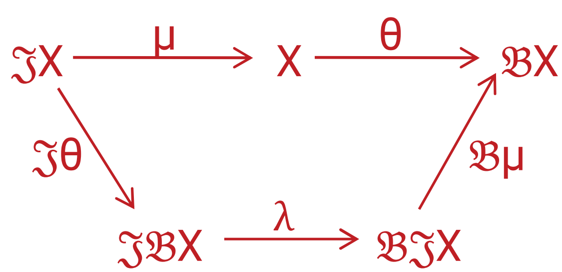
Nondeterministic Process Algebras

A successful solution for the nondeterministic case => Process Algebras (PAs)

- The concurrent communicating systems are conceptualised along two axes:
  - the behaviours are described by (labelled) transition systems (LTS) => coalgebras;
  - the compositionality is solved by construction principles => algebraic structure;
  - the algebraic and the coalgebraic structures are not independent:
    structural operational semantics (SOS) relates the two by defining the behaviour of a bigger system from the behaviours of its modules => bialgebra.

Two endofunctors $\mathfrak{B}$ (for behavior) and $\mathfrak{I}$ (for compositional specifications)

the class X of systems is simultaneously a **$\mathfrak{B}$-coalgebra** and a **$\mathfrak{I}$-algebra**.



- λ (a natural transformation between $\mathfrak{I}$ and $\mathfrak{B}$) defines a GSOS $\Longrightarrow$ **$\mathfrak{I}\mathfrak{B}$-Bialgebra**

D. Turi, G. Plotkin, *Towards a mathematical operational semantics*, LICS'97

Nondeterministic Process Algebras

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- PAs reflect elegant mathematical structures and are supported by easy and appealing underlying theories.

- PAs are based on a general framework for operational description of processes (GSOS) that makes the languages easy to extend with new (algebraic) operators.

- The underlying theory of PAs is robust: it can be canonically adapted when new algebraic operators are required => high applicability in various fields.

The challenge of stochastic processes

- The success of nondeterministic Process Algebras has inspired the research in the field of stochastic systems => Stochastic Process Algebras (SPAs)

- LTS is replaced by (labelled) Markovian processes (e.g., CTMCs)
  - nondeterministic a-transition:

$$P \xrightarrow{a} Q$$

  - stochastic (Markovian) a-transition:

$$P \xrightarrow{a,r} Q$$

    $r \in [0,+\infty)$ is the rate of an exponentially distributed random variable that characterises the a-transitions from P to Q.

- In recent decades a plethora of SPAs appeared, such as
  - TIPP (Gotz, Herzog, Rettelbach)
  - PEPA (Hillston)
  - EMPA (Bernardo, Gorrieri)
  - Stochastic pi-calculus (Priami, Degano)
  - StoKlaim (De Nicola, Katoen, Latella, Loreti, Massink)

  etc.

The challenge of stochastic processes

- In SPAs the nondeterminism is replaced by a "race policy" and this requires major modifications at the level of the underlying theory.
- The attempts to provide a pointwise semantics, similar to nondeterministic PAs, face the *counting problems* and the known SPAs solve them using "ad hoc" solutions such as the multi-transition system (PEPA) or the proved SOS (stochastic pi-calculus).

Problems

(B. Klin, V. Sassone, *Structural Operational Semantics for Stochastic Process Calculi*, FOSSACS'08)

- These SOS formalisms are difficult to extend to a general format for well-behaved stochastic specifications;
- In stochastic pi-calculus (with proved SOS) parallel composition is not associative up to bisimulation;
- In PEPA, if arbitrary relations between transition rates and the rates of subprocesses are allowed, stochastic bisimulation is not a congruence;

A possible explanation (ibid.): in a well-behaved SOS framework the labels of transitions should only carry as much data as required for the derivation of the intended semantics;

Both the proofs and the transition multiplicities contain superfluous data.

The challenge of stochastic processes

A solution to return to the simplicity and elegance of nondeterministic PAs:

instead the pointwise semantics, use a semantics based on distributions.

$$P \xrightarrow{a,r} Q \quad \Rightarrow \quad P \longrightarrow \mu, \quad \mu(a)(\{Q\})=r$$

where μ is a distribution (indexed by actions) on the measurable space of processes.

Similar approaches

R. Segala, N. Lynch, *Probabilistic Simulations for Probabilistic processes*, 1995.

M. Kwiatkowska, G. Norman, R. Segala, J. Sproston, *Automatic Verification of Real-Time Systems with Discrete Probability Distributions*, 1999.

E. P. de Vink, J. Rutten, *Bisimulation for probabilistic transition systems: A coalgebraic approach*, 1999.

J. Rutten, *Universal Coalgebra: a theory of systems*, 2000.

F. Bartels, *On Generalised Coinduction and Probabilistic Specification Formats*, 2004.

M. Bravetti, H. Hermanns, J.-P. Katoen, *YMCA: Why Markov Chain Algebra?*, 2006.

B. Klin, V. Sassone, *Structural Operational Semantics for Stochastic Process Calculi*, 2008.

R. De Nicola, D. Latella, M. Loreti, M. Massink, *Rate-based Transition Systems for Stochastic Process Calculi*, 2009.

In general, the space of processes considered is ($\mathbf{P}$, $2^{\mathbf{P}}$) – a compact representation of the pointwise semantics.

The role of Structural Congruence

Our approach: Structural congruence organizes a measurable space of processes; instead of $2^{\mathbf{P}}$ we use the sigma algebra Π generated by $\mathbf{P}^{\equiv}$; we use distributions on (**P**,Π).

Some of the advantages of our approach:

- Structural congruence is particularly appropriate for applications in Systems Biology

(it has been invented from chemical analogy G. Berry, G. Boudol, *The Chemical Abstract Machine*, 1990);

- By considering the distributions on (**P**,Π), the counting problems are solved on block while the labelles of the transition are the observable actions, as in the case of nondeterministic PAs;

- SOS is elegant, compact and the algebraic and coalgebraic structures are clean: SOS does not involve a rule of type (Struct); stochastic bisimulation is a congruence and it extends the structural congruence;

- The approach can be extended to other calculi; a general format can be defined, on the line of GSOS (Turi-Plotkin) or SGSOS (Klin-Sassone);

- The approach, extended to pi-calculus, provides a simple solution to the problems of bound output and replication (Cardelli, Mardare, *Stochastic Pi-Calculus revisited*, http://lucacardelli.name/)

- The approach allows a simple extension to metric semantics.

Stochastic CCS: The syntax

Let **A** be a denumerable set of action names endowed with

- an involution $*:\mathbf{A} \longrightarrow \mathbf{A}$, $a^* \neq a$, $a^{**}=a$
- a weight function $\iota:\mathbf{A} \longrightarrow \mathbb{Q}^+$, $\iota(a)=\iota(a^*)$ for all $a \in \mathbf{A}$

Let $\text{ƈ} \notin \mathbf{A}$ and $\mathbf{A}^+ = \mathbf{A} \cup \{\text{ƈ}\}$

The set **P** of processes are defined by the following grammar, for arbitrary $r \in \mathbb{Q}^+$.

$$P := 0 \mid \varepsilon.P \mid P|P \mid P+P$$

$$\varepsilon := a \in \mathbf{A} \mid \text{ƈ}(r)$$

We extend the weight function ι by $\iota(\text{ƈ}(r))=r$.

Structural Congruence "≡" is the smallest equivalence relation on **P** that satisfies

I. 1. $P|Q \equiv Q|P$; 2. $(P|Q)|R \equiv P|(Q|R)$; 3. $P|0 \equiv P$.

II. 1. $P+Q \equiv Q+P$; 2. $(P+Q)+R \equiv P+(Q+R)$; 3. $P+0 \equiv P$.

III. if $P \equiv Q$, then for any ε and any $R \in \mathbf{P}$,

1. $P|R \equiv Q|R$; 2. $P+R \equiv Q+R$; 3. $\varepsilon.P \equiv \varepsilon.Q$.

The measurable space

For arbitrary $P \in \mathbf{P}$, let $P^{\equiv}$ be the $\equiv$-equivalence class of P and $\mathbf{P}^{\equiv}$ the set of $\equiv$-equivalence classes of processes.

Let Π be the sigma-algebra generated by $\mathbf{P}^{\equiv}$ over $\mathbf{P}$.

Theorem: $(\mathbf{P},\Pi)$ is a measurable space.

Let $\Delta(\mathbf{P},\Pi)$ denote the set of measurable functions on $(\mathbf{P},\Pi)$.

For $S, T \in \Pi$, let $S \mid T = \bigcup_{P \in S,\, Q \in T} (P|Q)^{\equiv}$ and $S_T = \bigcup_{P|R \in S,\, P \in T} R^{\equiv}$

Lemma: If $S, T \in \Pi$ and $P \in \mathbf{P}$, then $S \mid T$ and S_T are measurable sets.

Structural Operational Semantics

First challenge: define the operational semantics based on rules of type

$$P \longrightarrow \mu,$$

where $\mu: \mathbf{A}^{+} \longrightarrow \Delta(\mathbf{P},\Pi)$ is such that

for each $x \in \mathbf{A}^{+}$, $\mu(x) \in \Delta(\mathbf{P},\Pi)$ is a measure and

for each $\mathcal{S} \in \Pi$, $\mu(x)(\mathcal{S}) = r \in \mathbb{Q}^{+}$,

r is the rate of the x-transition from P to (elements of) $\mathcal{S}$.

Notice that $\mathcal{S}$ is not just any set, but a measurable set, e.g. $\mu(x)(\{Q\})$ is undefined.

More generally, the operational semantics encodes a function

$$\theta: \mathbf{P} \longrightarrow [\mathbf{A}^{+} \longrightarrow \Delta(\mathbf{P},\Pi)]$$

defined by $\theta(P) = \mu$ iff $P \longrightarrow \mu$.

In this way, a stochastic process $P \in \mathbf{P}$ is just a Markov process $(P,\mathbf{P},\Pi,\theta)$, where $(\mathbf{P},\Pi,\theta)$ is an $\mathbf{A}^{+}$ - Markov kernel.

Second challenge: define the operational semantics such that

behavioural equivalence of SPA processes = stochastic bisimulation of Markov processes

Structural Operational Semantics

The null process "0"

Intuition: for each measurable set $S \in \Pi$, and any action $x \in \mathbf{A}^+$,

$$0 \xrightarrow{x,0} S.$$

Let $\varpi : \mathbf{A}^+ \longrightarrow \Delta(\mathbf{P},\Pi)$ such that for any $x \in \mathbf{A}^+$,

$$\varpi(x)=\omega,$$

where $\omega \in \Delta(\mathbf{P},\Pi)$ is the null measure defined, for arbitrary $S \in \Pi$, by

$$\omega(S)=0.$$

The first SOS rule is

$$\text{(Null)} \quad \frac{}{0 \longrightarrow \varpi}$$

Structural Operational Semantics

The prefixing "ε.P"

Intuition: $a.P \xrightarrow{a,\iota(a)} P^{\equiv}$,

for any measurable set $\mathcal{S} \in \Pi$, with $P \notin \mathcal{S}$,

$a.P \xrightarrow{a,0} \mathcal{S}$,

for any measurable set $\mathcal{S} \in \Pi$ and any $x \neq a$,

$a.P \xrightarrow{x,0} \mathcal{S}$.

$ƈ(r).P \xrightarrow{ƈ,r} P^{\equiv}$,

for any measurable set $\mathcal{S} \in \Pi$, with $P \notin \mathcal{S}$,

$ƈ(r).P \xrightarrow{ƈ,0} \mathcal{S}$,

for any measurable set $\mathcal{S} \in \Pi$ and any $x \neq ƈ$,

$ƈ(r).P \xrightarrow{x,0} \mathcal{S}$.

Let $\left[{\varepsilon \atop P^{\equiv}}\right]: \mathbf{A}^{+} \longrightarrow \Delta(\mathbf{P},\Pi)$ such that for any $a \in \mathbf{A}$,

$$\left[{\varepsilon \atop P^{\equiv}}\right](a) = \begin{cases} D(\iota(\varepsilon),P^{\equiv}), & a=\varepsilon \\ \omega, & a\neq\varepsilon \end{cases} \qquad \left[{\varepsilon \atop P^{\equiv}}\right](ƈ) = \begin{cases} D(\iota(\varepsilon),P^{\equiv}), & \varepsilon\notin\mathbf{A} \\ \omega, & \varepsilon\in\mathbf{A} \end{cases}$$

The second SOS rule is

$$(\text{Guard}) \; \frac{}{\varepsilon.P \longrightarrow \left[{\varepsilon \atop P^{\equiv}}\right]}$$

Structural Operational Semantics

The nondeterministic choice "+"

Intuition: if for some measurable set $S \in \Pi$ and some action x

$$P \xrightarrow{x,r} S \quad \text{and} \quad Q \xrightarrow{x,s} S,$$

then, $$P+Q \xrightarrow{x,r+s} S.$$

Let $\oplus : \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+} \times \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+} \longrightarrow \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}$

such that for any $\mu,\mu' \in \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}$, any action x and any measurable set S,

$$(\mu \oplus \mu')(x)(S) = \mu(x)(S) + \mu'(x)(S)$$

The third SOS rule is

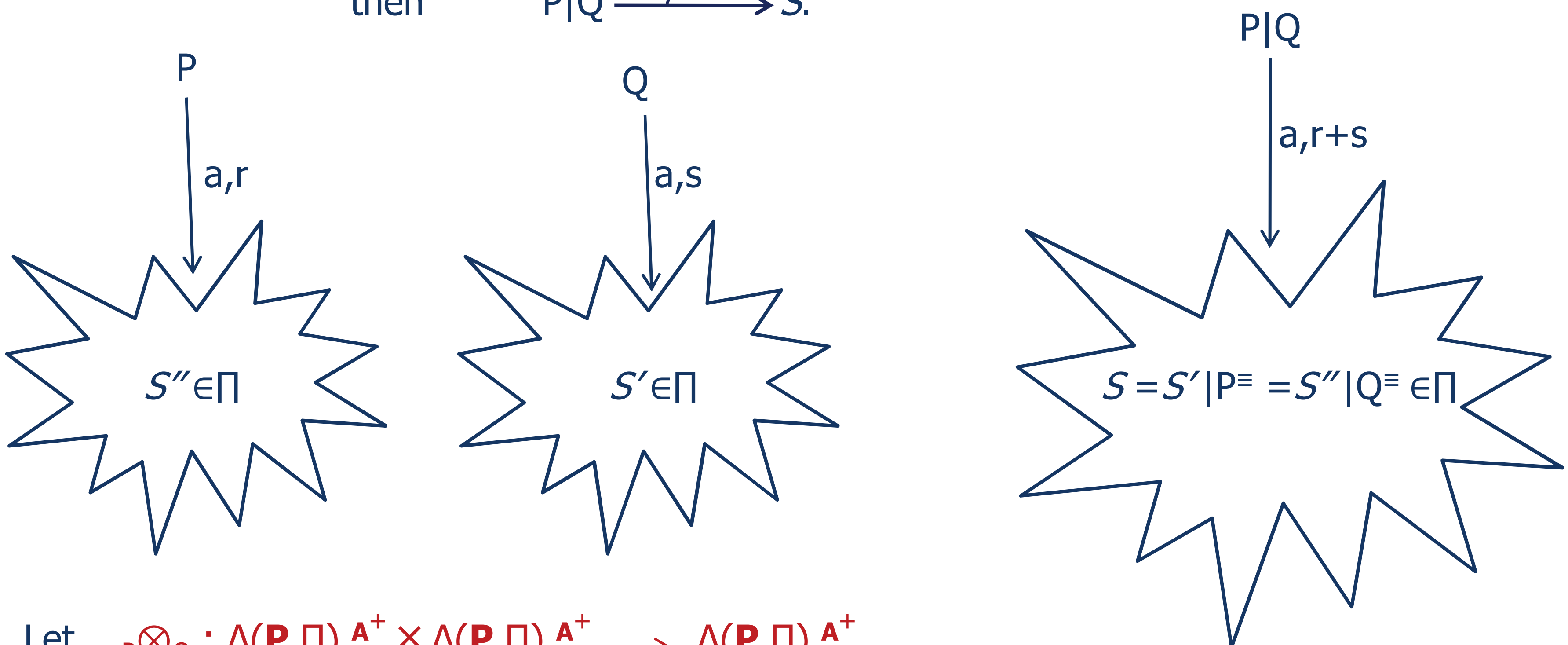
$$(\text{Sum}) \ \frac{P \longrightarrow \mu \qquad Q \longrightarrow \mu'}{P+Q \longrightarrow \mu \oplus \mu'}$$

Structural Operational Semantics

The parallel composition "|"

Intuition: for $a \in \mathbf{A}$, if $S = S' | P^{\equiv} = S'' | Q^{\equiv}$ and $P \xrightarrow{a,r} S''$, $Q \xrightarrow{a,s} S'$

then $P|Q \xrightarrow{a,r+s} S$.



Let $_{P}\otimes_{Q} : \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+} \times \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+} \longrightarrow \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}$

such that for any $\mu, \mu' \in \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}$, any atomic action a and any measurable set S,

$$(\mu \, _{P}\otimes_{Q} \, \mu')(a)(S) = \mu(a)(S_Q) + \mu'(x)(S_P)$$

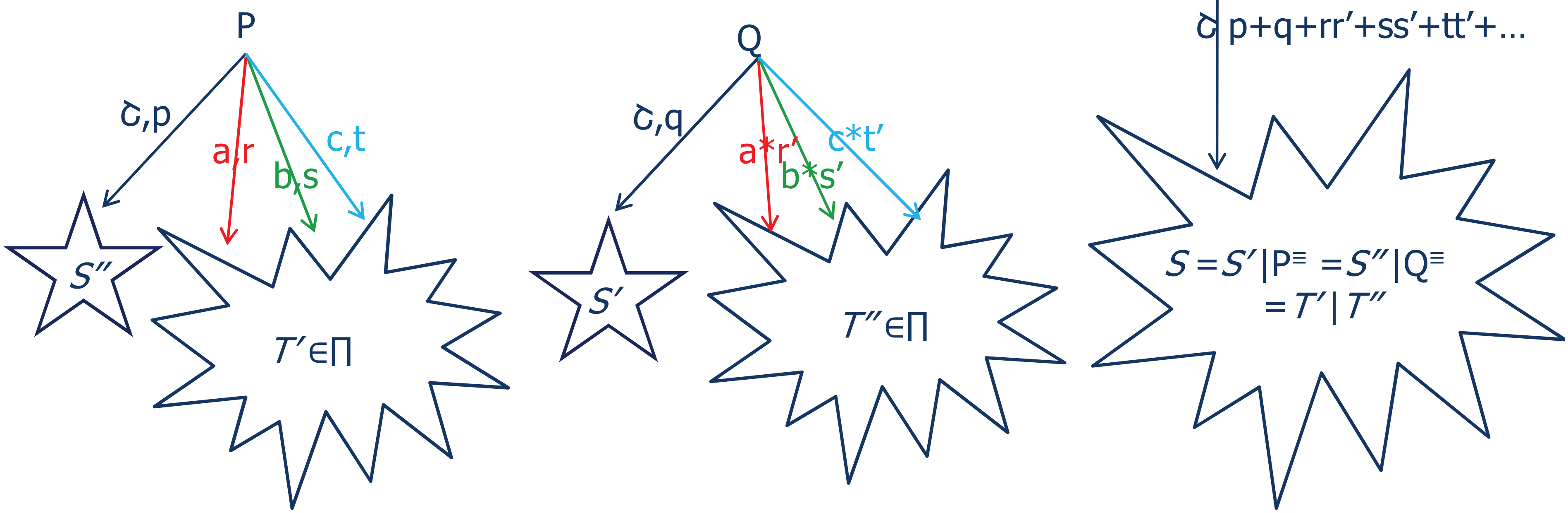
(if $P \longrightarrow \mu$ and $Q \longrightarrow \mu'$, then $P|Q \longrightarrow \mu \, _{P}\otimes_{Q} \, \mu'$)

Structural Operational Semantics

Intuition: for P|Q the rate of ↄ-action subsumes

- the rate of ↄ-action of P and the rate of ↄ-action of Q
- the rate of synchronizations between P and Q - we use the mass action law.

(M. Calder, S. Gilmore, J. Hillston, *Automatically deriving ODEs from process algebra models of signaling pathways*, CMSB'05.)



for μ, μ' with finite support:

$$(\mu\ {}_{P}\otimes_{Q}\ \mu')(ↄ)(S) = \mu(ↄ)(S_Q) + \mu'(ↄ)(S_P) + \sum_{T'|T''=S}^{a\in\mathbf{A}} \frac{\mu(a)(T')\times\mu'(a^*)(T'')}{2\iota(a)}$$

Structural Operational Semantics

The parallel composition "|"

For any $P,Q \in \Pi$, let $\;_P\otimes_Q : \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+} \times \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+} \longrightarrow \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}$
such that for any $\mu,\mu' \in \Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}$ (with finite support), any atomic action a and any $S \in \Pi$,

$$(\mu \;_P\otimes_Q \mu')(a)(S) = \mu(a)(S_Q) + \mu'(x)(S_P)$$

$$(\mu \;_P\otimes_Q \mu')(\text{ƈ})(S) = \mu(\text{ƈ})(S_Q) + \mu'(\text{ƈ})(S_P) + \sum_{T'|T''=S}^{a\in\mathbf{A}} \frac{\mu(a)(T') \times \mu'(a^*)(T'')}{2\iota(a)}$$

The fourth SOS rule is

$$\text{(Par)}\quad \frac{P \longrightarrow \mu \qquad Q \longrightarrow \mu'}{P|Q \longrightarrow \mu \;_P{}^{\equiv}\otimes_Q{}^{\equiv}\mu'}$$

The algebra of mappings

We have defined an algebraic structure $(\Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}, \varpi, [{}^{\varepsilon}_{P}], \oplus, {}_{P}\otimes_{Q})$ with operators defined for arbitrary ε, *P* and *Q*.

Lemma:

I. (1) $\mu \oplus \mu' = \mu' \oplus \mu$,

(2) $(\mu \oplus \mu') \oplus \mu'' = \mu \oplus (\mu' \oplus \mu'')$,

(3) $\mu \oplus \varpi = \mu$.

II. (1) $\mu \;{}_{P}\otimes_{Q}\; \mu' = \mu' \;{}_{Q}\otimes_{P}\; \mu$,

(2) $(\mu \;{}_{P}\otimes_{Q}\; \mu') \;{}_{P|Q}\otimes_{R}\; \mu'' = \mu \;{}_{P}\otimes_{Q|R}\; (\mu' \;{}_{Q}\otimes_{R}\; \mu'')$,

(3) $\mu \;{}_{P}\otimes_{0}\; \varpi = \mu$.

Notice that $(\mathbf{P}, 0, \varepsilon, +, |)$ and $(\Delta(\mathbf{P},\Pi)^{\mathbf{A}^+}, \varpi, [{}^{\varepsilon}_{P}], \oplus, {}_{P}\otimes_{Q})$ have different signatures.

Structural Operational Semantics

$$\text{(Null)}\quad \frac{}{0 \longrightarrow \varpi}$$

$$\text{(Guard)}\quad \frac{}{\varepsilon.P \longrightarrow [\,^{\varepsilon}_{P\equiv}]}$$

$$\text{(Sum)}\quad \frac{P \longrightarrow \mu \qquad Q \longrightarrow \mu'}{P+Q \longrightarrow \mu \oplus \mu'}$$

$$\text{(Par)}\quad \frac{P \longrightarrow \mu \qquad Q \longrightarrow \mu'}{P|Q \longrightarrow \mu\ {}_{P}{}^{\equiv}\!\otimes_{Q}{}^{\equiv}\mu'}$$

Lemma: For any $P \in \mathbf{P}$, there exists a unique $\mu \in \Delta(\mathbf{P},\Pi)^{\mathbf{A}^{+}}$ such that $P \longrightarrow \mu$. Moreover, μ has finite support.

Notice that we have no rule that guarantees that structural congruent processes have identical behaviour. But we can prove this.

Theorem: If $P \equiv Q$ and $P \longrightarrow \mu$, then $Q \longrightarrow \mu$.

Stochastic bisimulation

We can define θ: **P** $\longrightarrow$ [$\mathbf{A}^+ \longrightarrow \Delta(\mathbf{P},\Pi)$] by $\theta(P) = \mu$ iff $P \longrightarrow \mu$.

<u>Theorem:</u> ($\mathbf{P}$,Π,θ) is an $\mathbf{A}^+$ - Markov kernel and (P,$\mathbf{P}$,Π,θ) is a Markov process.

<u>Definition:</u> A *rate bisimulation* is an equivalence $\mathcal{R}$ on **P** such that for arbitrary $P,Q \in \mathbf{P}$ with $P \longrightarrow \mu$ and $Q \longrightarrow \mu'$, $(P,Q) \in \mathcal{R}$ iff for any $\mathcal{S} \in \Pi(\mathcal{R})$ and any $x \in \mathbf{A}^+$,

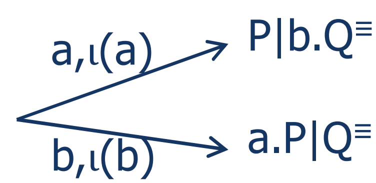
$$\mu(x)(\mathcal{S}) = \mu'(x)(\mathcal{S}).$$

The stochastic bisimilation is the reunion of rate bisimulations. Notation: $P \backsim Q$.

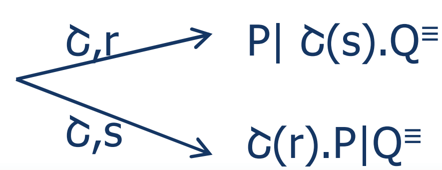
For arbitrary $P \in \mathbf{P}$, let $P^{\backsim}$ be the $\backsim$-equivalence class of P and $\mathbf{P}^{\backsim}$ the set of $\backsim$-equivalence classes of processes.

Examples (discussed in the paper)

1. $a,b \in \mathbf{A}$, $a^* \neq b$, $a.P|b.Q \backsim a.(P|b.Q)+b.(a.P|Q)$



2. $r \neq s$, ƨ(r). P | ƨ(s).Q ∽ ƨ(r).(P| ƨ(s).Q) + ƨ(s).(ƨ(r).P|Q)



Stochastic Bisimulation

Examples (discussed in the paper)

3. We have seen that for $a^* \neq b$, $a.0|b.0 \sim a.b.0+b.a.0$ => $\mathcal{S} = a.0|b.0^{\sim} = a.b.0+b.a.0^{\sim}$

Let $P = \tau(r).(a.0|b.0) + \tau(r).(a.b.0+b.a.0)$

$Q = \tau(r).(a.0|b.0) + \tau(r).(a.0|b.0)$

$R = \tau(r).(a.b.0+b.a.0) + \tau(r).(a.b.0+b.a.0)$

Observe that

$P \xrightarrow{\tau,2r} \mathcal{S}$ $\quad Q \xrightarrow{\tau,2r} \mathcal{S}$ $\quad R \xrightarrow{\tau,2r} \mathcal{S}$

and $P \sim Q \sim R$.

the three processes do not agree on any transition:

$P \xrightarrow{\tau,r} a.0|b.0$ $\quad Q \xrightarrow{\tau,2r} a.0|b.0$ $\quad R \xrightarrow{\tau,0} a.0|b.0$

$P \xrightarrow{\tau,r} a.b.0+b.a.0$ $\quad Q \xrightarrow{\tau,0} a.b.0+b.a.0$ $\quad R \xrightarrow{\tau,2r} a.b.0+b.a.0$

Lemma:

For arbitrary $P,Q \in \mathbf{P}$, if $P \equiv Q$, then $P \backsim Q$.

The reverse is not true, as shown in a previous example:

$a.0|b.0 \backsim a.b.0+b.a.0$ and $a.0|b.0 \not\equiv a.b.0+b.a.0$.

Theorem:

Stochastic bisimulation is a congruence, i.e.,

1. if $P \backsim P'$, then for arbitrary ε, $\varepsilon.P \backsim \varepsilon.P'$;
2. if $P \backsim P'$ and $Q \backsim Q'$, then $P+P' \backsim Q+Q'$;
3. if $P \backsim P'$ and $Q \backsim Q'$, then $P|P' \backsim Q|Q'$.

An Application: Metrics for Stochastic Processes

Bisimulation is a strict concept: it only verifies if two processes have identical behaviour.

It is useful to have a metric that measure the similarity of processes in terms of behaviours.

Our presentation of stochastic processes is particularly appropriate to define such a metric via metrics for distributions (e.g. Kantorovich metrics):

The intuition: the distance between P and Q, when $P \longrightarrow \mu$ and $Q \longrightarrow \mu'$, is given by

$$d(P,Q) = \sup_{x \in \mathbf{A}^+} \delta(\mu(x), \mu'(x))$$

where δ is a distance on distributions.

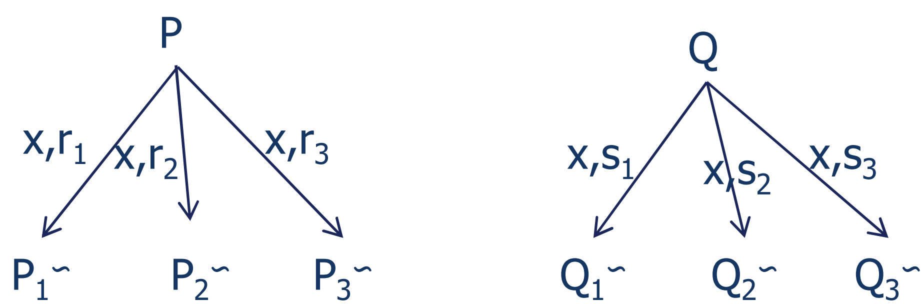
Related works on metrics for systems:

P. Lincoln, J. Mitchell, M. Mitchell, A. Scedrov, *A probabilistic poly-time framework for protocol analysis*, 1998
J. Desharnais, *Labelled Markov Processes*, 1999
F. Van Breugel, J. Worrell, *An Algorithm for Quantitative Verification of Probabilistic Systems*, 2001
L. de Alfaro, T. Henzinger, R. Majumdar, *Discounting the Future in Systems Theory*, 2003
V. Gupta, R. Jagadeesan, P. Panangaden, *Approximate Reasoning for Real-Time Probabilistic Processes*, 2006

An Application: Metrics for Stochastic Processes

A discount metric: let $c \in [0,1]$ and $x \in \mathbf{A}^+$; we define the pseudometric

$$d^c_x : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{R}^+$$



$$\begin{aligned} d^c_x(P,Q) = \min \{ \; & |r_1-s_1| + c\, d^c_x(P_1,Q_1) + |r_2-s_2| + c\, d^c_x(P_2,Q_2) + |r_3-s_3| + c\, d^c_x(P_3,Q_3), \\ & |r_1-s_1| + c\, d^c_x(P_1,Q_1) + |r_2-s_3| + c\, d^c_x(P_2,Q_3) + |r_3-s_2| + c\, d^c_x(P_3,Q_2), \\ & |r_1-s_2| + c\, d^c_x(P_1,Q_2) + |r_2-s_1| + c\, d^c_x(P_2,Q_1) + |r_3-s_3| + c\, d^c_x(P_3,Q_3), \\ & |r_1-s_2| + c\, d^c_x(P_1,Q_2) + |r_2-s_3| + c\, d^c_x(P_2,Q_3) + |r_3-s_1| + c\, d^c_x(P_3,Q_1), \\ & |r_1-s_3| + c\, d^c_x(P_1,Q_3) + |r_2-s_1| + c\, d^c_x(P_2,Q_1) + |r_3-s_2| + c\, d^c_x(P_3,Q_2), \\ & |r_1-s_3| + c\, d^c_x(P_1,Q_3) + |r_2-s_2| + c\, d^c_x(P_2,Q_2) + |r_3-s_1| + c\, d^c_x(P_3,Q_1) \; \} \end{aligned}$$

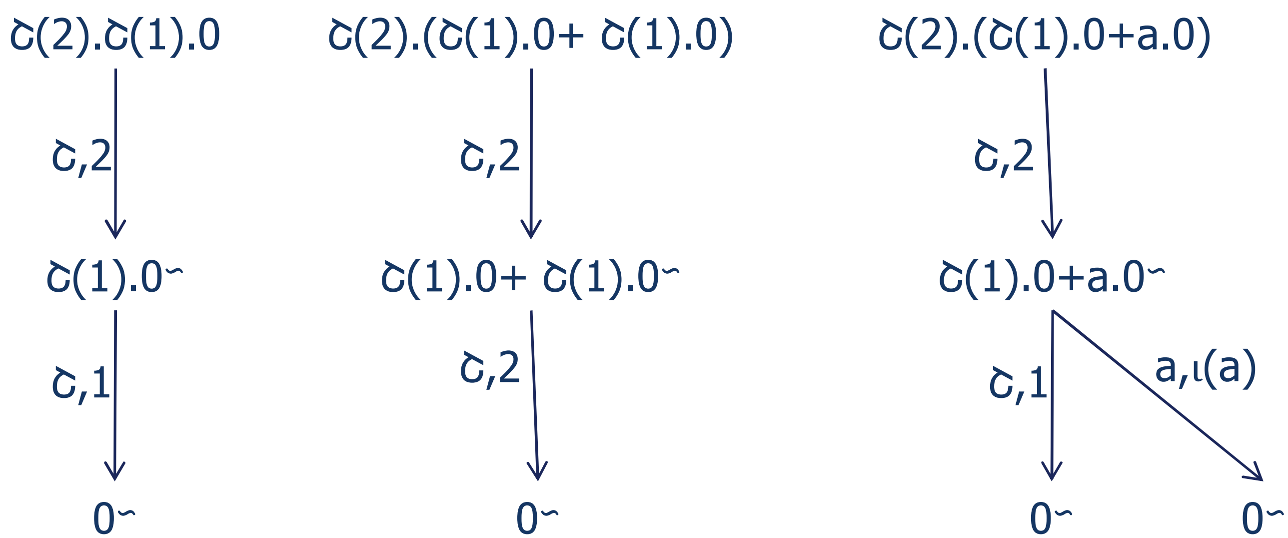
$$d^c : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{R}^+$$

$$d^c(P,Q) = \sup_{x \in \mathbf{A}^+} d^c_x(P,Q)$$

Notice that c discounts the future; if we take c=1 the future states count as the present state; if we take c=0 only the first step of the computation is considered.

An Application: Metrics for Stochastic Processes

Example (discussed in the paper)



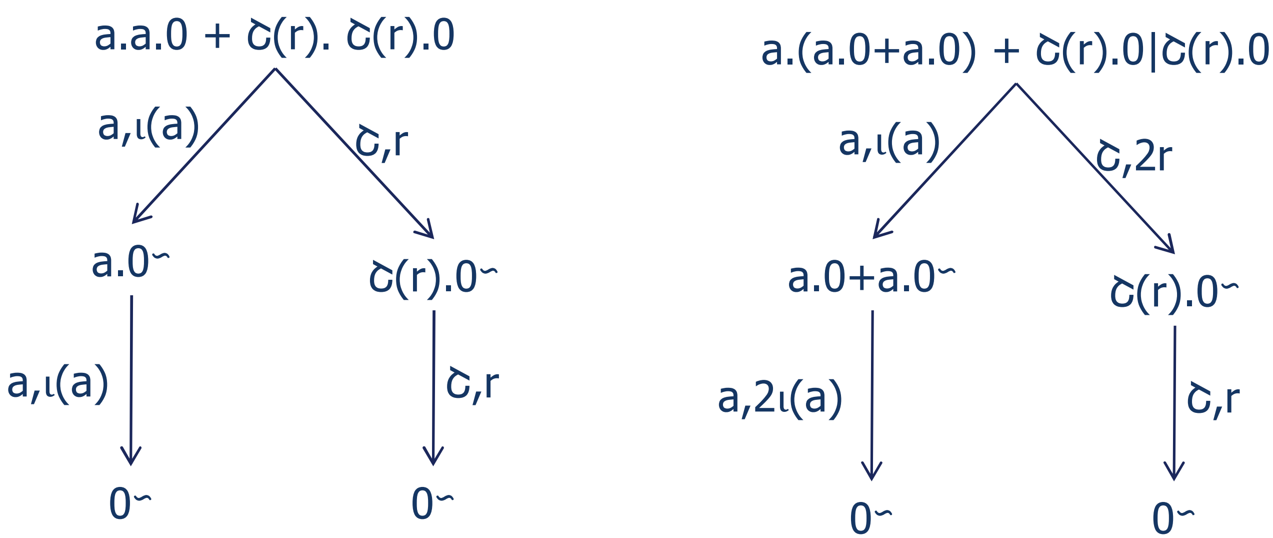
$d^{c}_{ƈ}$(ƈ(2).ƈ(1).0 , ƈ(2).(ƈ(1).0+ ƈ(1).0)) = |2-2| + c |2-1| = c

$d^{c}_{ƈ}$(ƈ(2).ƈ(1).0 , ƈ(2).(ƈ(1).0+a.0)) = |2-2| + c |1-1| = 0

$d^{c}_{ƈ}$(ƈ(2).(ƈ(1).0+ ƈ(1).0), ƈ(2).(ƈ(1).0+a.0))= |2-2| + c |2-1| = c

An Application: Metrics for Stochastic Processes

Example (discussed in the paper)



d^c(a.a.0 + ƈ(r). ƈ(r).0 , a.(a.0+a.0) + ƈ(r).0|ƈ(r).0) = max {c ι(a), r}

Conclusions

- We took the challenge of reconsidering Stochastic Process Algebras from a foundational perspective
- The goals:
  - understanding if the "ad hoc" approaches with their heavy mathematics can be avoided
  - providing well-behaved SOS formats similar to the formats of nondeterministic PAs
- The way to do it:
  - instead of trying to mimic the pointwise semantics of PAs, mimic their mathematical structures – move from the space of processes to the space of distributions
  - center the work on the equational theory of structural congruence => work with the equational monad instead of the freely generated monad
  - lift the algebraic structure from the space of processes to the space of distributions
- Advantages:
  - an elegant and compact SOS
  - well-behaved SOS: bisimulation is a congruence that extends structural congruence
  - a simple extension to metric semantics
  - simple solutions to the problems related to recursion and bound output
- The current state of our research:

  L. Cardelli, R. Mardare, *Stochastic Pi-Calculus Revisited*, *http://lucacardelli.name/*