An Imperative Object Calculus

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Outline

Object calculi are formalisms at the same level of abstraction as λ-calculi, but based exclusively on objects rather than functions.

- An untyped object calculus.
- An imperative operational semantics.
- A type system.
  - Self types.
  - Variance annotations.
  - Structural subtyping assumptions.
  - Polymorphism.
- Classes and inheritance.
- Typing soundness, based on store typings.

New from last year:

- Imperative/operational semantics (instead of functional/denotational).
- Primitive Self type (instead of encoded).
- Primitive variance annotations (instead of encoded).
- Structural subtyping assumptions (which are not denotationally sound).
- Soundness based on subject reduction (rather than models).
- Class encodings, requiring polymorphism and structural subtyping assumptions.

Syntax and Informal Semantics

The evaluation of terms is based on an imperative operational semantics with a global store; it proceeds deterministically from left to right.

Syntax of terms

\[
\begin{align*}
  a,b &::= \text{term} \\
  x &::= \text{variable} \\
  \lambda x \cdot \xi(x)b_i \mid i \in 1..n &::= \text{method invocation} \\
  a.l &::= \text{method update (imperative)} \\
  \text{let } x = a \text{ in } b &::= \text{let (sequential evaluation)} \\
  \text{clone}(a) &::= \text{cloning (shallow copy)}
\end{align*}
\]

- An object is a collection of components \( l = \xi(x)b_i \) for distinct labels (method names) \( l \) and associated methods \( \xi(x)b_i \). The methods are parameterless; \( x \) is a name for self within \( b_i \).
- The letter \( \xi \) (sigma) is a binder; it delays evaluation of the term to its right.

The let and method update constructs may be combined into a single construct, for more expressive typing (see FASE proceedings).
A Small Example

We define a memory cell with get, set, and dup (duplicate) components:

```
get = false,  
set = \(\zeta\)self \(\lambda\)b  
sel.set := b,  
set = \(\zeta\)self  
field field update
```

Some new constructions are used here:

- Procedures (\(\lambda\)), which can be encoded.
- Booleans, which can be encoded much as in the \(\lambda\)-calculus.
- Fields and field update, which can be desugared as follows:

```
let y_1 = false
in \{get = \zeta(self) y_1,
   set = \zeta(self) \lambda(b)
   let y_2 = b sel.set = \zeta(self) y_2,
   ...
\}
```

Procedures

Consider an imperative call-by-value \(\lambda\)-calculus that includes abstraction, application, and assignment to \(\lambda\)-bound variables. E.g.: \((\lambda(x) x:=x+1)(3)\) is a term yielding 4.

Translation of procedures

- \(\{x\}_p \triangleq p(x)\) if \(x \in\text{dom}(p)\), and \(x\) otherwise
- \(\{x:=a\}_p \triangleq x.arg:=\langle a\rangle_p\)
- \(\{\lambda(x)b\}_p \triangleq \langle \arg = \zeta(x)z.arg,\n   \text{val} = \zeta(x)\langle b\rangle[p := x.arg]\rangle\)
- \(\{b(a)\}_p \triangleq \langle \text{clone}(\langle b\rangle_p), arg := \langle a\rangle_p\rangle.val\)

Low-level interpretation

- The translation of a procedure \(\lambda(x)b\) is a stack frame with an uninitialized (divergent) argument slot (\(\text{arg}\)), and an initial program counter (\(\text{val}\)) that points to code accessing the argument slot through a frame pointer (\(x\)).
- The translation of a procedure call allocates a fresh stack frame (by \(\text{clone}\)), fills the argument slot (by :=), and jumps to the code (by .val).

Operational Semantics

The semantics relates terms to results in a global store.

Notation

- Store location (e.g., an integer)
- Object result (\(l\), distinct)
- Store for closures (\(i\), distinct)
- Stack for results (\(x\), distinct)

Well-formed store judgment:

\(\sigma \vdash o\)

Well-formed stack judgment:

\(\sigma;S \vdash o\)

Term reduction judgment:

\(\sigma;S \vdash a \rightarrow \nu;\sigma'\)

Sample rules

(Red Object) (\(l_i, i\), distinct)

\(\sigma;S \vdash o \quad \nu \in \text{dom}(\sigma) \quad \forall i \in 1..n\)

\(\sigma;S \vdash [l_i = \zeta(x) b_i \in\text{1..n}] \rightarrow [l_i = \zeta(x) b_i S] \in\text{1..n}]\)

(Red Select)

\(\sigma;S \vdash a \rightarrow [l_i = i \in\text{1..n}] \cdot \sigma' \quad \sigma'(i) = \zeta(x) b_i S) \quad \nu \in \text{dom}(\sigma') \quad \forall j \in 1..n\)

\(\quad \sigma' \cdot S \cdot x_i = [l_i = i \in\text{1..n}] \cdot b_i \rightarrow \nu;\sigma''\)

\(\sigma;S \vdash a, l \rightarrow \nu;\sigma''\)

(Red Simple Update)

\(\sigma;S \vdash a \rightarrow [l_i = i \in\text{1..n}] \cdot \sigma' \quad \nu \in \text{dom}(\sigma') \quad \forall j \in 1..n\)

\(\sigma;S \vdash a, l, p := \zeta(x) b \rightarrow [l_i = i \in\text{1..n}] \cdot \sigma' \cdot i_j = \zeta(x) b, S\)

N.B. The term:

\([[l = \zeta(x) x := x]_l]]\)

creates a loop in the store. An attempt to read out the result by "inlining" the store and stack mappings would produce the infinite term:

\([l = \zeta(x) [l = \zeta(x)]_l]\)
A Type System

We develop a type system for the imperative calculus. We treat Self types, variance annotations, and structural subtyping assumptions. Simpler (and less expressive) type systems could also be defined.

Syntax of types

<table>
<thead>
<tr>
<th>A, B ::=</th>
<th>type variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>the biggest type</td>
</tr>
<tr>
<td>Top</td>
<td>object type (υi ∈ I*)</td>
</tr>
<tr>
<td>Obj(X)[l: B; X]</td>
<td></td>
</tr>
</tbody>
</table>

Well-formed environment judgment: \( E \vdash \circ \)
Well-formed type judgment: \( E \vdash A \)
Subtyping judgments: \( E \vdash A \ll B \) \( E \vdash A \ll \nu \ B \)
Term typing judgment: \( E \vdash a : A \)

Self Types

Intent: memory cells can be typed as:

\( \text{MemDup} \triangleq \text{Obj}(X)[\text{get}: \text{Bool}, \text{set}: \text{Bool} \rightarrow X, \text{dup}: X] \)

In general, let:

\( A = \text{Obj}(X)[l: B; X] \triangleq \nu \)

- \( A \) is the type of those objects with methods named \( l \) and result types \( B[l[A] \).
- The binder \( \text{Obj} \) binds a Self type named \( X \) (which is known to be a subtype of \( A \)).

Moreover:
- The \( \nu \) are variance annotations.
- The variable \( X \) may occur only covariantly in the types \( B \).

Notation

- \( B[X] \) means that \( X \) may occur free in \( B \).
- \( B[X'] \) means that \( X \) occurs covariantly in \( B \).
- \( B[A] \) is the result of substituting \( A \) for \( X \) in \( B[X] \), where \( X \) is clear from context.

Note: Counterexample

\( \text{memDup} : \text{MemDup} \triangleq \)

\[ \begin{align*}
\text{memDup} & \triangleq \text{Obj}(X)[\text{get}: \text{Bool}\text{.set}(\text{false})\text{.dup in} \text{false}, \\
& \quad \text{set} = \xi(\text{self}) \lambda (b) \text{self}, \\
& \quad \text{dup} = \xi(\text{self}) \text{self} ]
\end{align*} \]

\( \text{mem} : \text{Mem} \triangleq \text{memDup} \) since \( \text{MemDup} \ll \text{Mem} \)

\( \text{mem.set} := \lambda (b) \xi(\text{false})\text{.get in} \text{false,} \\
\quad \text{set} = \xi(\text{self}) \lambda (b) \text{self} ] \\
\text{mem} = \)

\[ \begin{align*}
\text{mem.get} & \quad \text{FAILS!}
\end{align*} \]

The subtyping rule for object types with Self asserts, as usual, that a “longer” object type is a subtype of a “shorter” one.

A simplified rule for object types without variance annotations reads:

\( \begin{align*}
E, X \ll \text{Top} & \vdash B[X'] \quad \forall \nu \text{free in } X + m \\
E & \vdash \text{Obj}(X)[l; B; X] \triangleq \nu \ll \text{Obj}(X)[l; B; X] \triangleq \nu
\end{align*} \)

For example:

\( \begin{align*}
\text{Mem} & \triangleq \text{Obj}(X)[\text{get}: \text{Bool}, \text{set}: \text{Bool} \rightarrow X] \\
\text{MemDup} & \triangleq \text{Obj}(X)[\text{get}: \text{Bool}, \text{set}: \text{Bool} \rightarrow X, \text{dup}: X] \\
\text{MemDup} & \ll \text{Mem}
\end{align*} \)

The type \( \text{Obj}(X)[...] \) can be viewed as a recursive type \( \mu(X)[...] \), but with differences in subtyping that are crucial for object-oriented applications. The subtyping rule above is unsound with recursive types instead of Self types (i.e. with \( \mu \) instead of \( \text{Obj} \)), in presence of subsumption and update.
Variance Annotations

Again, let:

\[ A = \text{Obj}(X)[\forall u_i ; B_i[X] ^{\leq 1..n}] \]

Each \( u_i \) is a variance annotation; it is one of the symbols \( \checkmark \), \( \checkmark^\star \), and \( \checkmark^\circ \), for contravariance, invariance, and covariance, respectively.

Intuitively, \( \checkmark \) means read-only, \( \checkmark^\star \) means write-only, and \( \checkmark^\circ \) means read-write.

- \( \checkmark \) prevents update, but allows covariant component subtyping.
- \( \checkmark^\star \) prevents invocation, but allows contravariant component subtyping.
- \( \checkmark^\circ \) allows both invocation and update, but requires exact matching in subtyping.

By convention, any omitted \( u_i \)'s are taken to be equal to \( \checkmark^\circ \).

A simple object type:

\[ [i; B_i] ^{\leq 1..n} \]

is an abbreviation for \( \text{Obj}(X)[\forall u_i ; B_i[X] ^{\leq 1..n}] \), where \( X \) does not appear in any \( B_i \).

Variance Rules

Because of variance annotations, we use an auxiliary subtyping judgment:

\[
E,Y <: \text{Obj}(X)[\forall u_i ; B_i[X] ^{\leq 1..n}] \vdash u_i B_i[Y] <: u_i B_i[Y] \quad \forall i \in 1..n
\]

\[
E \vdash \text{Obj}(X)[\forall u_i ; B_i[X] ^{\leq 1..n}] <:_<: \text{Obj}(X)[\forall u_i ; B_i[X] ^{\leq 1..n}]
\]

- (Sub Invariant) An invariant component on the right requires an identical one on the left.
- (Sub Covariant) A covariant component type on the right cannot be a supertype of a corresponding component type on the left, either covariant or invariant. Intuitively, an invariant component can be regarded as covariant.
- (Sub Contravariant) A contravariant component type on the right can be a subtype of a corresponding component type on the left, either covariant or invariant. Intuitively, an invariant component can be regarded as contravariant.

Example: Procedure Types

A procedure with argument of type \( A \) and result of type \( B \), encoded as shown earlier, can be given type:

\[ [\text{arg}^{\checkmark^\circ} : A, \text{val}^{\checkmark^\circ} : B] \]

By the subtyping rules for variances we obtain:

\[ [\text{arg}^{\checkmark^\circ} : A, \text{val}^{\checkmark^\circ} : B] <: [\text{arg}^{\checkmark} : A, \text{val}^{\checkmark^\circ} : B] \]

By subsumption, any procedure has the type on the right. Therefore, we can take:

\[ A \rightarrow B \triangleq [\text{arg}^{\checkmark} : A, \text{val}^{\checkmark^\circ} : B] \]

Which yields a defined notion of procedure type that is contravariant in the argument and covariant in the result type.

Example: State Encapsulation

One can hide certain object components from view simply by subsumption; this technique can be used to encapsulating state.

Variance annotations enable more sophisticated forms of encapsulation.

\[
\text{Mem} \triangleq \text{Obj}(X)[\text{get} : \text{Bool}, \text{set} : \text{Bool} \rightarrow X]
\]

\[
\text{mem} : \text{Mem} \triangleq [\text{get} = \text{false}, \text{set} = \lambda (\text{self}) \lambda (b) \text{self} := b]
\]

When considering a memory cell as an object encapsulating state, it is natural to expect both components of \( \text{Mem} \) to be protected against external update. Take:

\[
\text{ProtectedMem} \triangleq \text{Obj}(X)[\text{get} : \text{Bool}, \text{set} : \text{Bool} \rightarrow X]
\]

Since \( \text{Mem} <: \text{ProtectedMem} \), any memory cell can be subsumed into \( \text{ProtectedMem} \) and thus protected against updating from the outside.

Note that the \( \text{set} \) method can still update the \( \text{get} \) field "from the inside".
Polymorphism

Additional syntax of terms

\[ a, b \quad ::= \quad \text{term} \]

\[ \ldots \quad \lambda b \quad \text{type abstraction} \]

\[ a() \quad \text{type application} \]

N.B. \( \lambda b \) is the type-erasur of \( \lambda (X:a)b \); \( a() \) is the type-erasur of \( a(A) \).

Additional results

\[ \psi \quad ::= \quad \text{result} \]

\[ \ldots \quad (\lambda (b,s)) \quad \text{type abstraction result} \]

Additional term reductions (…)

Additional syntax of types

\[ A, B \quad ::= \quad \text{type} \]

\[ \ldots \quad \forall (X:a)A \quad \text{bounded universal quantifier} \]

Additional typing rules (…)

Structural Subtyping Assumptions

(Val Field Update Non-Structural) \( (\text{where } A = \text{Obj}(X)[l_0\triangleright B_1(X) \rightarrow l^1,n]) \)

\[ E \vdash a : A \quad E, Y < A \vdash b : B[Y] \quad u \in \psi \quad \gamma \quad j \in 1..n \]

\[ E \vdash a.l_j = b : A \]

(Val Field Update) \( (\text{where } A' = \text{Obj}(X)[l_0\triangleright B_1(X) \rightarrow l^1,n]) \)

\[ E \vdash a : A \quad E \vdash A < A' \quad E, Y < A \vdash b : B[Y] \quad u \in \psi \quad \gamma \quad j \in 1..n \]

\[ E \vdash a.l_j = b : A \]

Mem \( \triangleq \text{Obj}(X)[\text{get}^*: \text{Bool}, \text{set}^*: \text{Bool} \rightarrow X] \)

\[ E, X < \text{Mem}, x; X, b; \text{Bool} \vdash x : X \]

\[ E, X < \text{Mem}, x; X, b; \text{Bool} \vdash x < \text{Mem} \]

\[ E, X < \text{Mem}, x; X, b; \text{Bool} \vdash x = \text{Bool} \]

\[ E, X < \text{Mem}, x; X, b; \text{Bool} \vdash x; \text{get} = b : X \]

\[ E, X < \text{Mem} \vdash \lambda (x) \lambda (b) x; \text{get} = b : \forall (X < \text{Mem}) X \rightarrow \text{Bool} \rightarrow X \]

N.B. We have obtained a non-trivial term of type \( \forall (X < \text{Mem}) X \rightarrow B[X] \). The non-structural rule would only yield \( \forall (X < \text{Mem}) X \rightarrow B[X] \).

Classes as Collections of Pre-Methods

We define classes as collections of reusable pre-methods.

- A pre-method is a procedure that is later used to construct a method.
- Each pre-method must work for all possible subclasses of a given class, so that it can be inherited and instantiated to any of these subclasses.
- To this end, pre-methods have types of the form \( \forall (X < a)X \rightarrow B_i[X] \).

We associate a class type \( \text{Class}(A) \) to each object type \( A \):

\[ \text{If } A \equiv \text{Obj}(X)[l_0\triangleright B_1(X) \rightarrow l^1,n] \]

\[ \text{then } \text{Class}(A) \triangleq \{ \text{new} : A, l_1; \forall (X < a)X \rightarrow B_i[X] \rightarrow l^1,n \} \]

The implementation of \( \text{new} \) is uniform for all classes: it produces an object of type \( A \) by collecting all the pre-methods and applying them to the sel of the new object.

\[ c : \text{Class}(A) \triangleq \{ \text{new} = \varphi(x) [l = \varphi(x) \downarrow l_1(x) \rightarrow l^1,n], l_1 = \ldots, l_n = \ldots \} \]

Classes

\[ \text{Class}(\text{Mem}) = \]

\[ \{ \text{new} : \text{Mem}, \]

\[ \text{get} : \forall (X < \text{Mem}) X \rightarrow \text{Bool}, \]

\[ \text{set} : \forall (X < \text{Mem}) X \rightarrow \text{Bool} \rightarrow X \]

\[ \text{memClass} : \text{Class}(\text{Mem}) \triangleq \]

\[ \{ \text{new} = \varphi(x) [\text{get} = \varphi(x) \downarrow \text{get}(0)(x), \text{set} = \varphi(x) \downarrow \text{set}(0)(x)], \]

\[ \text{get} = \lambda (x) \lambda (l) \text{false}, \]

\[ \text{set} = \lambda (x) \lambda (l) \lambda (b) x; \text{get} = b \}

\[ m : \text{Mem} \triangleq \text{memClass}.\text{new} \]

Note that the \( \text{set} \) pre-method receives the desired type (as shown earlier) thanks to the structural subtyping assumptions.
Subclasses and Inheritance

\[
\text{Class(MemDup)} \equiv \\
[\text{new: MemDup,} \\
\text{get: } \forall (x:\text{MemDup}) \ X \rightarrow \text{Bool,} \\
\text{set: } \forall (x:\text{MemDup}) \ X \rightarrow \text{Bool} \rightarrow X, \\
\text{dup: } \forall (x:\text{MemDup}) \ X \rightarrow X] \\
\text{memDupClass: Class(MemDup) } \triangleq \\
[\text{new = } \zeta(x) \ [\text{get = } \zeta(x) \ z.\text{get}(x), \text{set = } \zeta(x) \ z.\text{set}(x), \text{dup = } \zeta(x) \ z.\text{dup}(x)], \\
\text{get = memClass.get,} \\
\text{set = memClass.set,} \\
\text{dup = } \lambda(x) \ \lambda(x) \ \text{clone}(x)]
\]

Note that:
- memClass.set : \( \forall (x:\text{Mem}) X \rightarrow \text{Bool} \rightarrow X \)
- \( \forall (x:\text{Mem}) X \rightarrow \text{Bool} \rightarrow X \) \( \vdash \forall (x:\text{MemDup}) X \rightarrow \text{Bool} \rightarrow X \)
- by subsumption, memClass.set : \( \forall (x:\text{MemDup}) X \rightarrow \text{Bool} \rightarrow X \)
- therefore, memClass.set can be reused as a pre-method of Class(MemDup).

Soundness

Store types

\[
M ::= \text{Obj}(X)[l_{1} l_{2} B][X|^{l_{1}, l_{2}}] \rightarrow j \quad \text{method type } (j \in 1..n) \\
\Sigma ::= l_{1} \rightarrow M_{1} |^{l_{1}, l_{2}} \quad \text{store type } (l_{i} \text{ distinct})
\]

Type stacks

\[
T \equiv X_{p} \rightarrow A_{i} |^{l_{1}, l_{2}} \quad \text{type stack } (A_{i} \text{ closed types})
\]

Result typing judgment: \( \Sigma \vdash p : A \) (A closed)

Stack typing judgment: \( \Sigma \vdash S \vdash E \)

Store typing judgment: \( \Sigma \vdash \sigma \)

N.B. The fact that values are typed with respect to store types (and not stores) allows us to deal with cycles in the store.

Theorem (Subject Reduction)

If \( \theta \vdash a : A \) and \( \theta \vdash a \rightarrow \nu \sigma \)
then there exist a type \( A^{\dagger} \) and a store type \( \Sigma^{\dagger} \) such that
\( \Sigma^{\dagger} \vdash \sigma \) and \( \Sigma^{\dagger} \vdash \nu : A^{\dagger} \), with \( \theta \vdash A^{\dagger} \prec A \).

\[\Box\]

Conclusions

- We have described a basic calculus for imperative objects and their types.
- Because of its compactness and expressiveness, this calculus is appealing as a kernel for object-oriented languages that include subsumption and Self types.
- The calculus is not class-based, since classes are not built-in, nor delegation-based, since the method-lookup mechanism does not delegate invocations. However, the calculus models class-based languages well: classes and inheritance arise from object types and polymorphic types. In delegation-based languages, traits play the role of classes; our calculus can model traits just as easily as classes, along with dynamic delegation based on traits.