The Measurable Space of Stochastic Processes

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Abstract. We introduce a stochastic extension of CCS based on the mass action law and endowed with a structural operational semantics expressed in terms of measure theory. The measurable space of processes is defined for the σ-algebra of structural congruence-closed sets. The structural operational semantics associates to each process an action-indexed class of measures over the space of processes. These compute the rates of the transitions from a state of a system to a measurable set of states. In this setting, we prove that stochastic bisimulation is a congruence that extends structural congruence.

1 Introduction

Process algebras (PAs) [3] are formalisms designed for describing the evolution of concurrent communicating systems. For capturing observable behaviors, PAs are conceptualised along two orthogonal axes. From an algebraic point of view, they are endowed with construction principles in the form of algebraic operations that allow composing larger processes from more basic ones; a process is identified by its algebraic term. On the other hand, there exists a notion of nondeterministic evolution, described by a coalgebraic structure, in the form of a transition system. The algebraic and coalgebraic structures are not independent: Structural Operational Semantics (SOS) defines the behavior of a process inductively on its syntactic structure.

In the past decades *probabilistic* and *stochastic* behaviors have also become of central interest due to the applications in performance evaluation and computational systems biology. *Stochastic process algebras* such as TIPP [15], PEPA [17, 18], EMPA [4] and stochastic π-calculus [28] have been defined as extensions of classic PAs, by considering more complex coalgebraic structures. The label of a stochastic transition contains, in addition to the name of the action, the rate of an exponentially distributed random variable that characterizes the duration of the transition. Consequently, SOS associates a non-negative rate value to each tuple $\langle$*state, action, state*$\rangle$.

However, due to the additional quantitative information, the traditional approach to SOS is inadequate for modeling stochastic process calculi. The easy and appealing underlying theory that guaranteed the success of classic PAs becomes, in the stochastic case, heavy and problematic, requiring complicated variants of SOS, such as the *multi-transition system* approach of PEPA or the *proved SOS* approach of stochastic π-calculus mainly due to the fact that nondeterminism is replaced by the race policy.

With the intention of developing a stochastic process calculus for applications in computational systems biology, in this paper we propose a stochastic version of CCS [24] with mass action law [6], equipped with a non-standard operational semantics: the SOS rules are not given in the pointwise style, but using some constructions based on measure theory. We associate to each process a set of action-indexed measures on the measurable space of processes. Thus, for an action a, a process P and a measurable set S of processes, the measure μ_a associated to process P computes the rate $\mu_a(S) \in [0, \infty]$ of a-transitions from P to (elements of) S. In this way, difficult instance-counting problems that else require complicated versions of SOS can be solved by exploiting the properties of measures (e.g. additivity).

The same idea has been considered previously in literature. [22, 31] define the transitions of probabilistic automata by functions that associate to each state a discrete probability distribution over the state space. The *transition-systems-as-coalgebras* paradigm [9, 20, 30] exploits the same idea, providing a general mathematical characterisation of transition systems.

Similarly, in process algebras, extending the *general SOS* (GSOS) of [32] on stochastic settings, [21, 8] use measures on the *discrete measurable space of processes* to introduce the *stochastic general SOS* (SGSOS) and to study general congruence formats.

In this context, the novelty of our approach derives from the structure of the measurable space of stochastic processes that we consider. The algebra of processes is endowed in our case with *structural congruence* that is an *indistinguishability relation* on processes. In effect, the σ-algebra of processes contains only the structural congruence-closed sets. The reason for our choice is practical: structural congruence is the relation that equates terms representing processes that we do not differentiate from a modeling perspective. For instance, if we model the parallel evolution of two processes, say Q and R, we expect no difference between $Q|R$, $R|Q$ and $R|Q|0$ (where 0 denotes an inactive process). Consequently, if a process P can perform an action a with a rate r to give $Q|R$, written $P \xrightarrow{a,r} Q|R$, then we should also have $P \xrightarrow{a,r} R|Q$ and $P \xrightarrow{a,r} R|(Q|0)$. Because the rate is computed by a measure, which is an additive function, if any set of processes is measurable, then the transition $P \xrightarrow{a,3r} \{Q|R, R|Q, R|(Q|0)\}$ is legal – a fact with no practical basis. Similarly, if $Q|R \xrightarrow{a,r} P$, then we expect to also derive $R|Q \xrightarrow{a,r} P|0$ and $R|Q|0 \xrightarrow{a,r} (0|P)|0$.

While the SGSOS framework, as well as that of GSOS, focuses on the *monads freely generated* by the algebraic signature of the process calculus, we choose an *equational monad*: the structural congruence provides extra structure for the class of processes and thus we get a new type of SOS format. For instance, in our format the algebraic signature of processes is different from the algebraic signature of behaviors – all this is discussed in Remark 2, Section 6. Using a σ-algebra different from the powerset makes our approach more general than the others, while considering the measurable sets closed to some congruence relation makes it more appropriate for modeling and for extensions to other *equational theories*.

Central to our development is the concept of *stochastic transition kernel* (STK) which is a generalisation for arbitrary measurable spaces of the notion of a *rate transition system* [21, 8]. Inspired by similar concepts developed in probabilistic settings [20, 30, 9], STK lifts the notions of *Markov process* [5, 11, 10] and *Harsanyi type space* [16, 27] on to the stochastic level. We also introduce a notion of *stochastic bisimulation* for STKs, along the lines of probabilistic bisimulation [23, 11, 10, 12]. Our stochastic bisimulation generalizes *rate aware bisimulation* introduced in [8], being defined for arbitrary measurable spaces and closed with respect to an equational theory (defined by structural congruence). With respect to other stochastic process calculi where this notion is problematic (e.g., in some versions of stochastic π-calculus or PEPA parallel composition can fail to be associative up to stochastic bisimulation, see [21, 8]), we prove that for our calculus *stochastic bisimulation* is a congruence that extends structural congruence. This result completes the result of [21] where it is shown that for the calculi with CCS-like communication satisfying the mass action law (but without an equational theory) the parallel operator is associative up to stochastic bisimulation.

The paper is organized as follows. A preliminary section fixes the basic concepts and notations. Section 3 introduces the syntax of our minimal process algebra and the axiomatization of structural congruence. Sections 4 and 5 introduce the concepts of stochastic transition kernels and their bisimulation and then applies them to the space of processes. This allows us to introduce the operational semantics in Section 6. Section 7 is dedicated to the problem of stochastic bisimulation on stochastic processes. We also have a concluding section. The proofs of the results and the detailed discussions of the examples are collected in the Appendix.

2 Preliminaries

In this section we recall a few notions of measure theory to establish the terminology and the notations used in the paper.

For arbitrary sets A and B, 2^A denotes the powerset of A, $[A \to B]$ and B^A denote the class of functions from A to B. For an arbitrary function $f : A \to B$, the *kernel of* f is the relation $ker(f) = \{(x, y) \in A \times A \mid f(y) = f(x)\}$. As usual $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote the naturals, rationals and reals, respectively.

Given a set M, a *σ-algebra* Σ over M is a set of subsets of M containing M and closed under complement and countable union. Observe that 2^M is a σ-algebra. The tuple (M, Σ) is called a *measurable space* and the elements of Σ *measurable sets*. A measurable space (M, Σ) is called *discrete*[1], if $\Sigma = 2^M$.

[1] Textbooks do not agree on the definition of discrete measurable spaces regarding the nature of the support set. In this paper we chose a general definition used, for instance, in [13].

Given a measurable space (M, Σ), a *base* of it is a set $\Omega \subseteq \Sigma$ having the following properties.

1). For any $(N_i)_{i\in I} \subseteq \Omega$, $I \subseteq \mathbb{N}$ and $N_i \neq N_j$ for $i \neq j$, $\cup_{i\in I} N_i \notin \Omega$.

2). For each $S \in \Sigma$, there exists[2] $(N_i)_{i\in I} \subseteq \Omega$, $I \subseteq \mathbb{N}$, such that $S = \cup_{i\in I} N_i$.
In this case, we say that Σ is the *σ-algebra generated by Ω* and that (M, Σ) is the *measurable space generated by Ω*.

Given a set M, any denumerable partition S of M is a base for (M, Σ), where

$$\Sigma = \{\bigcup_{i\in I} s_i \mid \text{ for arbitrary } s_i \in S, I \subseteq \mathbb{N}\}.$$

Given a measurable space $\mathcal{M} = (M, \Sigma)$, a *measure on $\mathcal{M}$* is a function $\mu : \Sigma \to [0, \infty]$ such that

1) $\mu(\emptyset) = 0$,

2) if $(N_i)_{i\in I}$ is a set of pairwise disjoint sets in Σ, $I \subseteq \mathbb{N}$, then

$$\mu(\bigcup_{i\in I} N_i) = \sum_{i\in I} \mu(N_i).$$

We denote by $\Delta(M, \Sigma)$ the class of measures on (M, Σ).

The *null measure* on (M, Σ) is the measure ω such that $\omega(M) = 0$.

Notice that if (M, Σ) is a measurable space generated by Ω, any function $f : \Omega \to [0, \infty]$ defines a unique measure on (M, Σ). For this reason we define, for $N \in \Omega$ and $r \in [0, \infty]$, the *r-Dirac measure on N* that is the measure induced by the function $f : \Omega \to [0, \infty]$ defined as follows.

$$f(N') = \begin{cases} r \text{ if } N' = N \\ 0 \text{ if } N' \neq N \end{cases}$$

When (M, Σ) and Ω are fixed, we use r_N to denote the r-Dirac measure on N.

3 A minimal Stochastic Process Algebra

In this section we introduce a stochastic extension of CCS without replication [24]. As usual in these calculi, each action a used to label a transition has associated a *rate* (in the interval $[0, \infty]$) representing the parameter of an *exponentially distributed random variable* that characterizes the duration of an a-synchronization. As in CCS, the set of actions is equipped with an *involution* that associate to each action a its paired action $\overline{a}$. The rates characterize the synchronizations of the paired actions following the *mass action law* [6].

The next definition introduces the structured set of actions.
Let $\mathbb{Q}_+^\infty = (\mathbb{Q} \cap (0, \infty)) \cup \{\infty\}$.

Definition 1 (Bipartite Weighted Set). *A* bipartite weighted set *is a countable set* $\mathbb{A}$ *endowed with*

[2] Notice that $\emptyset \subseteq \mathbb{N}$, $\emptyset = (N_i)_{i\in\emptyset} \subseteq \Omega$ and $\emptyset = \cup_{i\in\emptyset} N_i$.

• *an* involution $\overline{} : \mathbb{A} \to \mathbb{A}$*, i.e. a function that associates to each* $a \in \mathbb{A}$ *an element* $\overline{a} \in \mathbb{A}$ *such that* $a \neq \overline{a}$ *and* $\overline{\overline{a}} = a$*;*

• *a* weight function $\iota : \mathbb{A} \to \mathbb{Q}_+^\infty$*, such that for any* $a \in \mathbb{A}$*,* $\iota(a) = \iota(\overline{a})$*.*

A partial bipartite weighted set $\mathbb{B}$*, is a set such that* $\mathbb{B} = \mathbb{A} \cup \{\tau_{(r)} \mid r \in \mathbb{Q}_+^\infty\}$*, where* $\mathbb{A}$ *is a bipartite weighted set and* $\tau \notin \mathbb{A}$*.*

We say that $\mathbb{B}$ is a partial extension of $\mathbb{A}$. We also extend the weight function from $\mathbb{A}$ to $\mathbb{B}$, $\iota : \mathbb{B} \to \mathbb{Q}_+^\infty$, by $\iota(\tau_{(r)}) = r$. For $r \in \mathbb{Q}_+^\infty$, let $\mathbb{A}_r = \{a \in \mathbb{A} \mid \iota(a) = r\}$.

In this paper we are interested in the partial extensions of a bipartite set $\mathbb{A}$ up to the choice of $\tau \notin \mathbb{A}$. For this reason, given $\mathbb{A}$, we assume that there is a unique partial extension of it and we denote it by $\mathbb{A}^+$. We use a, a', a_i to denote arbitrary elements of $\mathbb{A}$ and $\alpha, \alpha', \alpha_i$ to denote arbitrary elements of $\mathbb{A}^+$.

The role of τ is to represent *internal actions*. We can have different internal actions resulting from synchronizations of actions with different rates. For this reason, the internal actions are indexed by their rates. Thus, $\tau_{(r)}$ represents a synchronization with rate r. The internal actions will be used as labels of the operational semantics and also in the syntax of our calculus for modeling delays.

Definition 2 (Stochastic Processes). *Let* $\mathbb{A}^+$ *be a partial bipartite weighted set. The* $\mathbb{A}^+$-stochastic processes *are defined, on top of a constant* 0 *and for arbitrary* $\alpha \in \mathbb{A}^+$*, inductively as follows.*

$$P := 0 \;\vdots\; \alpha.P \;\vdots\; P|P \;\vdots\; P + P.$$

We denote by $\mathbb{P}$ the set of stochastic processes.

Essential for processes is *structural congruence relation* which equates processes that, in spite of their different syntactic representation, are practically used to model the same systems.

Definition 3 (Structural congruence). *Structural congruence is the smallest relation* $\equiv \subseteq \mathbb{P} \times \mathbb{P}$ *that satisfies the following conditions.*

I. $(\mathbb{P}, |, 0)$ *is a commutative monoid for* $\equiv$*, i.e., for arbitrary* $P, Q, R \in \mathbb{P}$*,*

1. $P|Q \equiv Q|P$*;* 2. $(P|Q)|R \equiv P|(Q|R)$*;* 3. $P|0 \equiv P$*.*

II. $(\mathbb{P}, +, 0)$ *is a commutative monoid for* $\equiv$*, i.e., for arbitrary* $P, Q, R \in \mathbb{P}$*,*

1. $P + Q \equiv Q + P$*;* 2. $(P + Q) + R \equiv P + (Q + R)$*;* 3. $P + 0 \equiv P$*.*

III. $\equiv$ *is an equivalence relation on* $\mathbb{P}$*, i.e., for arbitrary* $P, Q, R \in \mathbb{P}$*,*

1. $P \equiv P$*;*
2. *if* $P \equiv Q$*, then* $Q \equiv P$*;*
3. *if* $P \equiv Q$ *and* $Q \equiv R$*, then* $P \equiv R$*.*

IV. $\equiv$ *is a congruence with respect to the algebraic structure of* $\mathbb{P}$*, i.e., for arbitrary* $P, Q, R \in \mathbb{P}$ *and* $\alpha \in \mathbb{A}^+$*,*

1. *if* $P \equiv Q$*, then* $P|R \equiv Q|R$*;*
2. *if* $P \equiv Q$*, then* $P + R \equiv Q + R$*;*
3. *if* $P \equiv Q$*, then* $\alpha.P \equiv \alpha.Q$*.*

Being an equivalence relation, $\equiv$ divides $\mathbb{P}$ in equivalence classes. Let $\mathbb{P}^\equiv$ be the set of $\equiv$-equivalence classes on $\mathbb{P}$. For arbitrary $P \in \mathbb{P}$, we denote by $P^\equiv$ the $\equiv$-equivalence class of P.

Remark 1. $\mathbb{P}^{\equiv}$ is a denumerable partition of $\mathbb{P}$, hence it generates a σ-algebra Π over $\mathbb{P}$; thus, $(\mathbb{P}, \Pi)$ is a measurable space. The measurable sets are (possibly denumerable) reunions of $\equiv$-equivalence classes on $\mathbb{P}$. In what follows we use $\mathcal{P}, \mathcal{P}_i, \mathcal{R}, \mathcal{Q}$ to denote arbitrary measurable sets of Π.

For arbitrary $\mathcal{P}, \mathcal{Q} \in \Pi$ and arbitrary $P \in \mathbb{P}$, consider

$$\mathcal{P}|\mathcal{Q} = \bigcup_{P \in \mathcal{P}, Q \in \mathcal{Q}} (P|Q)^{\equiv} \quad \text{and} \quad \mathcal{P}_P = \bigcup_{P|R \in \mathcal{P}} R^{\equiv}.$$

Notice that $\mathcal{P}|\mathcal{Q}$ and $\mathcal{P}_P$ are measurable sets, i.e., $\mathcal{P}|\mathcal{Q}, \mathcal{P}_P \in \Pi$.

4 Stochastic transition kernels

Before proceeding with the presentation of the operational semantics for the stochastic process algebra, we introduce the notion of *stochastic transition kernel* (STK) that is central to the future developments. This notion is inspired by the *transition-systems-as-coalgebras* paradigm [20, 30, 9] that defines a transition system by associating to each state of a system an action-indexed set of functions over the state space: functions with *boolean values* define *labelled transition systems* and *probabilistic distributions* define *Markov processes* [5, 11, 10] and *Harsanyi type spaces* [16, 27]. Similarly, an STK associates to each state a set of action-indexed measures over the measurable state space. Because we do not consider a necessarily discrete state space but an arbitrary measurable space, we also get a generalization of the concept of a *rate transition system* [21, 8].

Stochastic bisimulation of STKs is introduced along the lines of probabilistic bisimulation [23, 11, 7, 10, 12] and extends, for arbitrary measurable spaces, *rate aware bisimulation* introduced in [8].

Definition 4 (Stochastic transition kernels). *Given a measurable space (M, Σ) and a set $\mathbb{A}$ (of* labels*), an $\mathbb{A}$-*stochastic transition kernel *is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where $\theta : M \to [\mathbb{A} \to \Delta(M, \Sigma)]$.*

STKs describe the stochastic behavior of a system. If $m \in M$ is the current state, $a \in \mathbb{A}$ is the label of an action and $N \in \Sigma$ is a measurable set of states, $\theta(m)(a)$ is a measure and $\theta(m)(a)(N) \in [0, \infty]$ represents the total rate of the a-transitions from m to arbitrary $n \in N$.

Given a binary relation $\mathfrak{R} \subseteq M \times M$ on a set M, we call a subset $N \subseteq M$ $\mathfrak{R}$-closed iff $\{m \in M \mid \exists n \in N, (n, m) \in \mathfrak{R}\} \subseteq N$.
If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

Definition 5 (Bisimulations on stochastic transition kernels). *Given an $\mathbb{A}$-stochastic transition kernel (M, Σ, θ), a* rate-bisimulation relation *on it is an equivalence relation $\mathfrak{R} \subseteq M \times M$ such that $(m, n) \in \mathfrak{R}$ iff for any $C \in \Sigma(\mathfrak{R})$ and any $a \in \mathbb{A}$,*

$$\theta(m)(a)(C) = \theta(n)(a)(C).$$

Two elements $m_1, m_2 \in M$ are stochastic bisimilar*, written $m_1 \sim m_2$, if they are related by a rate-bisimulation relation.*

Observe that, for any $\mathbb{A}$-stochastic transition kernel (M, Σ, θ), there exist rate-bisimulation relations. For instance, the identity of the elements of M, $=\subseteq M \times M$ is a rate-bisimulation relation on (M, Σ, θ), because $\Sigma(=) = \Sigma$ and if $m = n$, then for any $C \in \Sigma$ and $a \in \mathbb{A}$, $\theta(m)(a)(C) = \theta(n)(a)(C)$.

Notice also that, because a rate-bisimulation $\mathfrak{R}$ is an equivalence relation, $\Sigma(\mathfrak{R}) = \Sigma \cap 2^{M^{\mathfrak{R}}}$, where $M^{\mathfrak{R}}$ is the quotient of M by $\mathfrak{R}$.

5 The stochastic transition kernel of processes

In this section we organize the measurable space $(\mathbb{P}, \Pi)$ of stochastic processes as an $\mathbb{A}^+$-stochastic transition kernel, where Π is the σ-algebra generated by the quotient of $\mathbb{P}$ with structural congruence, as shown in Remark 1. This implicitly provides an operational semantics for the minimal stochastic process algebra.

Pivotal for the entire construction is the space $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ of the functions $\mu : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$. Definition 6 constructs, inductively on the structure of processes, a function $\theta : \mathbb{P} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ which organizes $(\mathbb{P}, \Pi, \theta)$ as an STK. The intuition is that for an arbitrary process P and arbitrary $\mathcal{P} \in \Pi$, $\theta(P)(\tau_{(r)})(\mathcal{P})$ represents the rate of $\tau_{(r)}$ actions from P to (elements of) $\mathcal{P}$; for arbitrary $a \in \mathbb{A}$, $\theta(P)(a)(\mathcal{P})$ counts the possibilities of P to perform an action a to (elements of) $\mathcal{P}$, this value eventually being used to calculate the rate of $\tau_{(\iota(a))}$ actions from P.

For condensing the next definition, we consider the following operation on reals $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $x * y = x + y$ if $x \neq 0$ and $y \neq 0$ and $x * y = 0$, else.

Definition 6. *Let $\theta : \mathbb{P} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ be the function defined, inductively on the structure of $P \in \mathbb{P}$, as follows.*
***The case** $P = 0$. For any $\alpha \in \mathbb{A}^+$, let $\theta(0)(\alpha) = \omega$, where $\omega \in \Delta(\mathbb{P}, \Pi)$ is the null measure.*
***The case** $P = \tau_{(r)}.Q$. Let $\theta(\tau_{(r)}.Q)(\tau_{(r)}) = r_{Q^{\equiv}}$, where $r_{Q^{\equiv}}$ is the r-Dirac measure[3] on $Q^{\equiv}$ and for any $\alpha \in \mathbb{A}^+ \setminus \{\tau_{(r)}\}$, let $\theta(\tau_{(r)}.Q)(\alpha) = \omega$.*
***The case** $P = a.Q$, $a \in \mathbb{A}$. Let $\theta(a.Q)(a) = 1_{Q^{\equiv}}$, where $1_{Q^{\equiv}}$ is the 1-Dirac measure on $Q^{\equiv}$ and for any $\alpha \in \mathbb{A}^+ \setminus \{a\}$, let $\theta(a.Q)(\alpha) = \omega$.*
***The case** $P = Q + R$. For any $a \in \mathbb{A}$, $\mathcal{P} \in \Pi$ and $r \in \mathbb{Q}_+^\infty$, let*

$$\theta(Q+R)(a)(\mathcal{P}) = \theta(Q)(a)(\mathcal{P}) + \theta(R)(a)(\mathcal{P}),$$

$$\theta(Q+R)(\tau_{(r)})(\mathcal{P}) = \sum_{r=s+t} [\theta(Q)(\tau_{(s)})(\mathcal{P}) * \theta(R)(\tau_{(t)})(\mathcal{P})].$$

[3] Notice that $\{Q^{\equiv}, Q \in \mathbb{P}\}$ is a base of Π, hence, for any $r \in [0, \infty]$ we can define the r-Dirac measure $r_{Q^{\equiv}}$ on arbitrary $Q^{\equiv}$.

The case $P = Q|R$***.*** *For any* $a \in \mathbb{A}$*,* $\mathcal{P} \in \Pi$ *and* $r \in \mathbb{Q}_+^\infty$*, let*[4]

$$\theta(Q|R)(a)(\mathcal{P}) = \theta(R)(a)(\mathcal{P}_Q) + \theta(Q)(a)(\mathcal{P}_R),$$

$$\theta(Q|R)(\tau_{(r)})(\mathcal{P}) = \theta(R)(\tau_{(r)})(\mathcal{P}_Q) + \theta(Q)(\tau_{(r)})(\mathcal{P}_R) +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{P}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \theta(Q)(a)(\mathcal{P}_1) \cdot \theta(R)(\overline{a})(\mathcal{P}_2)}{2}.$$

If we define the *set of active actions of a process* $P \in \mathbb{P}$ by $act(0) = \emptyset$, $act(\alpha.P) = \{\alpha\}$, $act(P + Q) = act(P|Q) = act(P) \cup act(Q)$, then any process has only a finite set of active actions. Notice that $\theta(P)(\alpha) \neq \omega$ iff $\alpha \in act(P)$. This means that for any $\alpha \notin act(P)$ and any $\mathcal{R} \in \Pi$, $\theta(P)(\alpha)(\mathcal{R}) = 0$. Consequently, the two infinitary sums involved in Definition 6, for the cases $P = Q + R$ and $P = Q|R$, have finite numbers of non-null summands.

The next lemma proves the correctness of Definition 6 stating that for each $P \in \mathbb{P}$ and each $\alpha \in \mathbb{A}^+$, $\theta(P)(\alpha) \in \Delta(\mathbb{P}, \Pi)$, i.e. it is a measure.

Lemma 1. $(\mathbb{P}, \Pi, \theta)$ *is an* $\mathbb{A}^+$*-stochastic transition kernel.*

In Definition 6, for organizing the class $\mathbb{P}$ of processes as an $\mathbb{A}^+$-stochastic transition kernel, we used some particular operations on the functions $\mu : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$. In what follows we analyze the mathematical structure and the properties of these operations, that will be eventually used for describing, in a uniform and compact way, the SOS rules for our process algebra.

Definition 7. *Let* $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *be the space of the functions* $\mu : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$*. Consider the following constants and operations on* $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$*.*

1. Let $\omega^{\mathbb{A}^+} \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *be the function* $\omega^{\mathbb{A}^+} : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ *such that for any* $\alpha \in \mathbb{A}^+$*,* $\omega^{\mathbb{A}^+}(\alpha) = \omega$ *(the null measure).*

2. For arbitrary $\alpha \in \mathbb{A}^+$ *and* $P \in \mathbb{P}$*, let* $\alpha_P \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *be the function* $\alpha_P : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ *such that*

$$\alpha_P(x) = \begin{cases} 1_{P^\equiv} & ,\text{if } x = \alpha, \alpha \in \mathbb{A} \\ \iota(\alpha)_{P^\equiv} & ,\text{if } x = \alpha, \alpha \notin \mathbb{A} \\ \omega & ,\text{if } x \neq \alpha \end{cases}$$

where $1_{P^\equiv}$ *and* $\iota(\alpha)_{P^\equiv}$ *are* 1*-Dirac and* $\iota(\alpha)$*-Dirac measures on* $P^\equiv$*, respectively, and* ω *is the null measure.*

3. For two arbitrary functions $\mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$*, let* $\mu' \oplus \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *be the function such that for any* $a \in \mathbb{A}$ *and any* $r \in \mathbb{Q}_+^\infty$*,*

$$(\mu' \oplus \mu'')(a) = \mu'(a) + \mu''(a),$$

$$(\mu' \oplus \mu'')(\tau_{(r)}) = \sum_{r=s+t} [\mu'(\tau_{(s)}) * \mu''(\tau_{(t)})].$$

[4] The summands of the infinitary sum are divided by 2 because we count the interaction rates twice, due to the fact that $a = \overline{\overline{a}}$ for any $a \in \mathbb{A}$.

4. For arbitrary $P, Q \in \mathbb{P}$*, let* ${}_P\!\otimes_Q : \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+} \times \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *such that for any* $a \in \mathbb{A}$*,* $r \in \mathbb{Q}_+^\infty$ *and* $\mathcal{R} \in \Pi$*,*

$$(\mu' \ {}_P\!\otimes_Q \ \mu'')(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}_Q) + \mu''(a)(\mathcal{R}_P) \ \textit{and}$$

$$(\mu' \ {}_P\!\otimes_Q \ \mu'')(\tau_{(r)})(\mathcal{R}) = \mu'(\tau_{(r)})(\mathcal{R}_Q) + \mu''(\tau_{(r)})(\mathcal{R}_P) +$$

$$+ \sum_{\mathcal{P}_1 | \mathcal{P}_2 \subseteq \mathcal{R}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu'(a)(\mathcal{P}_1) \cdot \mu''(\overline{a})(\mathcal{P}_2)}{2}.$$

Observe that the sum involved in the definition of $\oplus$ has a denumerable number of summands, as $r, s, t \in \mathbb{Q}_+^\infty$; also the sum involved in the definition of ${}_P\!\otimes_Q$ is denumerable.

The next lemma proves that the definitions of $\oplus$ and ${}_P\!\otimes_Q$ for arbitrary $P, Q \in \mathbb{P}$ are correct and it also states some basic properties of these operators. In general, we can speak of lifting the algebraic structure of the class $\mathbb{P}$ of processes to the class of functions (i.e. indexed measures) $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$. However, it is not an "authentic" lifting as the signature on $\mathbb{P}$ is not the same with the signature of $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$; for instance, to the operator $|$ from $\mathbb{P}$ corresponds a class of indexed operators ${}_P\!\otimes_Q$ in $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ and for fixed $P, Q \in \mathbb{P}$, ${}_P\!\otimes_Q$ is neither commutative nor associative. We will eventually speak of an authentic lifting in Section 7, where we prove that by taking the quotient of $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ with stochastic bisimulation we obtain the algebraic structure of $\mathbb{P}$.

Lemma 2. *1. If* $\mu, \mu' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *and* $P, Q \in \mathbb{P}$*, then* $\mu \oplus \mu'$*,* $\mu \ {}_P\!\otimes_Q \ \mu' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$*.*
2. $(\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}, \oplus, \omega^{\mathbb{A}^+})$ *is a commutative monoid, i.e., for any* $\mu, \mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$*,*
(a). $\mu \oplus \mu' = \mu' \oplus \mu$*,* *(b).* $(\mu \oplus \mu') \oplus \mu'' = \mu \oplus (\mu' \oplus \mu'')$*,* *(c).* $\mu = \mu \oplus \omega^{\mathbb{A}+}$*.*
3. For arbitrary $\mu', \mu'', \mu''' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ *and arbitrary* $P, Q, R \in \mathbb{P}$*,*
(a). $\mu' \ {}_P\!\otimes_Q \ \mu'' = \mu'' \ {}_Q\!\otimes_P \ \mu'$*,*
(b). $(\mu' \ {}_P\!\otimes_Q \ \mu'') \ {}_{P|Q}\!\otimes_R \ \mu''' = \mu' \ {}_P\!\otimes_{Q|R} \ (\mu'' \ {}_Q\!\otimes_R \ \mu''')$*,*
(c). $\mu' \ {}_P\!\otimes_0 \ \omega^{\mathbb{A}^+} = \mu'$*.*
4. For arbitrary $\mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$*,* $P, P', Q, Q' \in \mathbb{P}$*, and* $\alpha \in \mathbb{A}^+$
(a.) if $P \equiv P'$ *and* $Q \equiv Q'$*, then* $\mu' \ {}_P\!\otimes_Q \ \mu'' = \mu' \ {}_{P'}\!\otimes_{Q'} \ \mu''$*,*
(b). if $P \equiv Q$*, then* $\alpha_P = \alpha_Q$*.*

6 Structural Operational Semantics

Using the operations defined in the previous section on the space of functions $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$, we introduce the structural operational semantics for the minimal stochastic process algebra that induces on $(\mathbb{P}, \Pi)$ the structure of an $\mathbb{A}^+$-stochastic transition kernel described in Lemma 1. The structural operational semantics associates to each process $P \in \mathbb{P}$ a function from $\mathbb{A}^+$ to $\Delta(\mathbb{P}, \Pi)$, i.e., a class of measures on $(\mathbb{P}, \Pi)$ indexed by the elements of $\mathbb{A}^+$. This mapping can be used to define both the *unlabeled operational semantics* (concerning only the

τ actions and their rates) and the *labeled operational semantics* (concerning all the actions in $\mathbb{A}^+$ and their rates).

The rule-schemas of the structural operational semantics, given for arbitrary $P, Q \in \mathbb{P}$ and $\alpha \in \mathbb{A}^+$, are listed in the next table. The *stochastic transition relation* is the smallest relation $\rightarrow \subseteq \mathbb{P} \times \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ satisfying these rules.

Reduction Rules

$$(Null).\ \frac{}{0 \rightarrow \omega^{\mathbb{A}^+}} \qquad (Sum).\ \frac{P \rightarrow \mu' \qquad Q \rightarrow \mu''}{P + Q \rightarrow \mu' \oplus \mu''}$$

$$(Guard).\ \frac{}{\alpha.P \rightarrow \alpha_P} \qquad (Par).\ \frac{P \rightarrow \mu' \qquad Q \rightarrow \mu''}{P|Q \rightarrow \mu' \ {}_P\!\otimes_Q \mu''}$$

Lemma 3. *If $P \equiv Q$ and $P \rightarrow \mu$, then $Q \rightarrow \mu$.*

The previous lemma guarantees that the SOS does not differentiate the structural congruent processes. Notice that the relation $\rightarrow$ is defined inductively on the syntactic structure of the processes and it associates a unique function $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ to each $P \in \mathbb{P}$. This means that we can associate to $\rightarrow$ a function $\theta : \mathbb{P} \rightarrow \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ defined, for arbitrary $P \in \mathbb{P}$, by

$$\theta(P) = \mu \text{ iff } P \rightarrow \mu.$$

Remark 2. There exist differences between our SOS format and the GSOS format of [32] or the SGSOS format proposed in [21]. In our case, the algebraic signature of the processes differs from the algebraic signature of the class of functions (behaviors); moreover, to the parallel operator "$|$" corresponds, in the domain of functions, a denumerable class of binary operators indexed by processes, i.e. "${}_P\!\otimes_Q$"! This non-standard situation is a consequence of the fact that $\equiv \subsetneq ker(\theta)$. If we consider the processes $P = a.0|b.0$ and $Q = a.b.0 + b.a.0$ for $a, b \in \mathbb{A}$ and $\{a, \overline{a}\} \cap \{b, \overline{b}\} = \emptyset$, then $P \rightarrow \mu_1 = [a_0 \ {}_{a.0}\!\otimes_{b.0} b_0]$ and $Q \rightarrow \mu_2 = [a_{b.0} \oplus b_{a.0}]$. It is trivial to verify that $\mu_1 = \mu_2$, however $P \not\equiv Q$. This shows that for some $R \rightarrow \nu$, we can have $\mu_1 \ {}_P\!\otimes_R \nu \neq \mu_1 \ {}_Q\!\otimes_R \nu$. Hence, due to the parallel operator and to the σ-algebra we have chosen, it is not possible to provide an SOS that uses the same signature for processes and for behaviors (functions) as in the classic case. However, in Section 7, after introducing the stochastic bisimulation relation "$\sim$" over processes and over functions, we will see that by taking the quotient of $\sim$ on both domains we will get the same algebraic signatures, meaning that we eventually have a "well-behaved" SOS, but up to stochastic bisimulation.

With the definition of θ, we recover the structure $(\mathbb{P}, \Pi, \theta)$ of $\mathbb{A}^+$-stochastic transition kernel described in Lemma 1. For each process $P \in \mathbb{P}$ the mapping $\theta(P) : \mathbb{A}^+ \rightarrow \Delta(\mathbb{P}, \Pi)$ is such that for each $\equiv$-closed set of processes $\mathcal{P} \in \Pi$, $\theta(P)(\tau_{(r)})(\mathcal{P}) \in [0, \infty]$ represents the total rate of the $\tau_{(r)}$-reductions of P to some arbitrary element of $\mathcal{P}$. We can use this to derive the classic pointwise semantics by defining $P \xrightarrow{\tau_{(r)}, s} Q$ iff $\theta(P)(\tau_{(r)})(Q^{\equiv}) = s$.

Since we consider that only the τ-actions (the interactions) have rates, for $a \in \mathbb{A}$, $\theta(P)(a)(\mathcal{P})$ counts the available a-actions that P can perform to (elements of) $\mathcal{P}$; this value, together with $\iota(a)$, is used to calculate the total rates of the $\tau_{(\iota(a))}$-actions from $C[P]$ to $C[\mathcal{P}]$, where $C[\_]$ is an arbitrary context of P.

Alternatively, one can use our SOS to define a labeled semantics for the stochastic calculus that associates rates not only to τ-actions, but to any action $a \in \mathbb{A}$ by $P \xrightarrow{a,s} Q$ iff $s = \iota(a) \cdot \theta(P)(a)(Q^{\equiv})$.

In what follows, for arbitrary $\mathcal{R} \in \Pi$, we use the notation $P \xrightarrow{\tau_{(r)},s} \mathcal{R}$ for $\theta(P)(\tau_{(r)})(\mathcal{R}) = s$ and $P \xrightarrow{a,s} \mathcal{R}$ for the case when $\iota(a) \cdot \theta(P)(a)(\mathcal{R}) = s$.

Example 1. Suppose that $a, b, c \in \mathbb{A}$ with $a, \overline{a}, b, \overline{b}, c, \overline{c}$ pairwise distinct and $\iota(a) = \iota(\overline{a}) = r$. It is trivial to check that

1. $a.P|a.P \xrightarrow{a,2r} a.P|P$,
2. $a.P|\overline{a}.Q \xrightarrow{\tau_{(r)},r} P|Q$,
3. $(a.P_1 + b.P_2)|(\overline{a}.Q_1 + c.Q_2) \xrightarrow{\tau_{(r)},r} P_1|Q_1$.

Lemma 4. *For an arbitrary process $P \in \mathbb{P}$, the following sets are finite*
$\{\alpha \in \mathbb{A}^+ \mid P \xrightarrow{\alpha,r} \mathbb{P}, r \neq 0\}$, $\{Q^{\equiv} \in \Pi \mid P \xrightarrow{\alpha,r} Q, \alpha \in \mathbb{A}^+, r \neq 0\}$.

7 Stochastic bisimulation is a congruence

This section is dedicated to the study of stochastic bisimulation for stochastic processes. In the pointwise approach, since the operational semantics requires various mathematical artifacts such as the multi-transition systems [17, 18] or the proved SOS [28, 29], the problem of stochastic bisimulation is difficult. Recently, an elegant solution was proposed in [21] but only for the case when there are no equational restrictions on the algebraic level. As argued before, for practical modeling purposes, our algebra is endowed with an equational theory of structural congruence that organizes the measurable space of processes and consequently stochastic bisimulation requires a different treatment.

In this section we introduce the stochastic bisimulation relying on the notion of bisimulation of stochastic transition kernels. This measure-theoretic approach helps to prove a more general result that agrees with the intuitions about the behavior of stochastic processes: the structural bisimulation on stochastic processes is a congruence that extends the structural congruence.

Before proceeding with the technical developments we informally recall that, in abstract algebra, given a set $\mathcal{X}$ with an algebraic structure, a *congruence relation* on $\mathcal{X}$ is an equivalence relation on $\mathcal{X}$ preserving its algebraic structure.

Stochastic bisimulation on processes is defined as stochastic bisimulation over the stochastic transition kernel $(\mathbb{P}, \Pi, \theta)$. We make this explicit in the next definition.

Definition 8 (Stochastic bisimulation on processes). *A* rate-bisimulation relation *on processes is an equivalence relation $\mathfrak{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$, $(P, Q) \in \mathfrak{R}$ iff for any $C \in \Pi(\mathfrak{R})$ and any $\alpha \in \mathbb{A}^+$,*

$$\mu(\alpha)(C) = \mu'(\alpha)(C).$$

Two processes $P, Q \in \mathcal{P}$ are stochastic bisimilar, written $P \sim Q$, iff there exists a rate bisimulation relation $\mathfrak{R}$ such that $(P, Q) \in \mathfrak{R}$.

The next theorem provides a characterization of stochastic bisimulation.

Theorem 1. *The stochastic bisimulation $\sim$ is the smallest equivalence relation on $\mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$, $P \sim Q$ iff for any $C \in \Pi(\sim)$ and any $\alpha \in \mathbb{A}$, $\mu(\alpha)(C) = \mu'(\alpha)(C)$.*

We denote by $\mathbb{P}^{\sim}$ the set of $\sim$-equivalence classes on $\mathbb{P}$, and for arbitrary $P \in \mathbb{P}$ we denote by $P^{\sim}$ the $\sim$-equivalence class containing P. We use the notation $P \not\sim Q$ to say that P and Q are not stochastic bisimilar.

Next we consider two examples. The first example shows that the stochastic bisimulation can differentiate processes that are bisimilar in CCS with reduction semantics. The second example shows that there exist stochastic bisimilar processes that are not structural congruent.

Example 2. For $a, b \in \mathbb{A}$, $a \neq b$ it is trivial to verify that
1. $a.0 \not\sim b.0$, 2. $a.0|b.0 \sim a.b.0 + b.a.0$.

The next lemma states that $\sim$ is an extension of the kernel of θ.

Lemma 5. *For arbitrary $P, Q \in \mathbb{P}$, if $P \to \mu$ and $Q \to \mu$, then $P \sim Q$.*

The next example shows that there exists no "pointwise" characterization of stochastic bisimulation similar with the one that characterizes bisimulation for labeled transition system.

Example 3. Let $b, c \in \mathbb{A}$ be two distinct actions. In Example 2 we showed that $b.0|c.0 \sim b.c.0+c.b.0$. Consider the processes $P = \tau_{(r)}.(b.0|c.0)+\tau_{(r)}.(b.c.0+c.b.0)$, $Q = \tau_{(r)}.(b.0|c.0) + \tau_{(r)}.(b.0|c.0)$ and $R = \tau_{(r)}.(b.c.0 + c.b.0) + \tau_{(r)}.(b.c.0 + c.b.0)$.

If C is the $\sim$-equivalence class that contains $b.0|c.0$ and $b.c.0 + c.b.0$, then $P \xrightarrow{\tau_{(r)},2r} C$, $Q \xrightarrow{\tau_{(r)},2r} C$, $R \xrightarrow{\tau_{(r)},2r} C$ and for any other $\sim$-equivalence class C', $P \xrightarrow{\tau_{(r)},0} C'$, $Q \xrightarrow{\tau_{(r)},0} C'$ and $R \xrightarrow{\tau_{(r)},0} C'$. Consequently, $P \sim Q \sim R$ (also because for any other action the rate is 0 everywhere). On the other hand,

$$P \xrightarrow{\tau_{(r)},r} b.0|c.0 \text{ and } P \xrightarrow{\tau_{(r)},r} b.c.0 + c.b.0,$$

$$Q \xrightarrow{\tau_{(r)},2r} b.0|c.0 \text{ and } Q \xrightarrow{\tau_{(r)},0} b.c.0 + c.b.0,$$

$$R \xrightarrow{\tau_{(r)},0} b.0|c.0 \text{ and } R \xrightarrow{\tau_{(r)},2r} b.c.0 + c.b.0.$$

Notice that they are not agreeing on any "pointwise" transition.

The previous example proves that the only definition for stochastic bisimulation, similar to the one of the bisimulation for labeled transition systems, can be given as follows.

$P \sim Q$ iff for arbitrary $\alpha \in \mathbb{A}^+$:

1. If $P \xrightarrow{\alpha,r_1} P_1, ..., P \xrightarrow{\alpha,r_n} P_n$ and $P_1, ..., P_n$ are all the processes R (up to structural congruence) such that $P \xrightarrow{\alpha,r} R$ for some $r \neq 0$, then there exist $Q_1, ..., Q_m$ such that $Q \xrightarrow{\alpha,s_1} Q_1, ..., Q \xrightarrow{a,s_m} Q_m$ and $Q_1, ..., Q_m$ are all the processes S (up to structural congruence) such that $Q \xrightarrow{a,s} S$ for some $s \neq 0$ and for each $\sim$-equivalence class C, if $P_{i_1}, ..., P_{i_k} \in C$ and $P_{i_{k+1}}, .., P_{i_n} \notin C$, then there exist $Q_{j_1}, ..., Q_{j_l} \in C$ such that $Q_{j_{1+1}}, ..., Q_{j_s} \notin C$ and $r_{i_1} + .. + r_{i_k} = s_{j_1} + .. + s_{j_l}$.
2. And reverse.

The relation $\sim$ on $\mathbb{P}$ can be lifted to $\Delta(P)^{\mathbb{A}^+}$ by defining, for arbitrary $\mu, \mu' \in \Delta(\mathbb{P})^{\mathbb{A}^+}$, $\mu \sim \mu'$ iff for any $C \in \mathbb{P}^{\sim}$ and any $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu'(\alpha)(C)$. Notice that $\sim \subseteq \Delta(\mathbb{P})^{\mathbb{A}^+} \times \Delta(\mathbb{P})^{\mathbb{A}^+}$ is an equivalence relation. We denote by $(\Delta(\mathbb{P})^{\mathbb{A}^+})^{\sim}$ the set of $\sim$-equivalence classes on $\Delta(\mathbb{P})^{\mathbb{A}^+}$ and for an arbitrary $\mu \in \Delta(\mathbb{P})^{\mathbb{A}^+}$ we denote by $\mu^{\sim}$ the $\sim$-equivalence class of μ.

With this notation, from Theorem 1 we derive the next corollary.

Corollary 1. *For arbitrary $P, Q \in \mathbb{P}$, if $P \rightarrow \mu$ and $Q \rightarrow \mu'$, then $P \sim Q$ iff $\mu \sim \mu'$.*

A consequence of $\sim$ being an equivalence on $\Delta(\mathbb{P})^{\mathbb{A}^+}$ is the next theorem that shows that our processes behave “correctly” with respect to structural congruence.

Theorem 2. *For arbitrary $P, Q \in \mathbb{P}$, if $P \equiv Q$, then $P \sim Q$.*

An immediate consequence of this theorem is that the parallel composition is associative with respect to stochastic bisimulation – a feature problematic in other stochastic process algebras. In [21] was proved that for the calculi using the mass action law the parallel composition is associative with respect to stochastic bisimulation, but the result did not refer to the calculi with an equational theory. This shows the novelty of the previous theorem.

In addition to the result of the previous theorem, notice that $\sim$ is strictly larger than $\equiv$, because for arbitrary $a, b \in \mathbb{A}$ we have $a.0|b.0 \sim a.b.0 + b.a.0$ and $a.0|b.0 \not\equiv a.b.0 + b.a.0$.

The next is the main theorem of this section stating that stochastic bisimulation is a congruence.

Theorem 3 (Congruence). *Stochastic bisimulation on $\mathbb{P}$ is a congruence relation with respect to the algebraic structure of $\mathbb{P}$, i.e. for arbitrary $P, P', Q, Q' \in \mathbb{P}$ and $\alpha \in \mathbb{A}^+$,*
1. if $P \sim P'$, then $\alpha.P \sim \alpha.P'$;
2. if $P \sim P'$ and $Q \sim Q'$, then $P + Q \sim P' + Q'$;
3. if $P \sim P'$ and $Q \sim Q'$, then $P|Q \sim P'|Q'$.

Because $ker(\theta) \subseteq \sim$ and $\sim$ is a congruence for processes, we deduce that if $P \sim P'$, $Q \sim Q'$, $P \rightarrow \mu$, $P' \rightarrow \mu'$, $Q \rightarrow \nu$ and $Q' \rightarrow \nu'$, then $\mu \;{}_P\!\otimes_Q \nu \sim \mu' \;{}_{P}\!\otimes_{Q'} \nu'$ and for any $\alpha \in \mathbb{A}^+$, $\alpha_P \sim \alpha_{P'}$. This shows that by taking the quotient with

$\sim$ both on processes and on functions, we will obtain identical signatures for processes and for behaviors (functions) and one could provide a SOS in the stile of [32, 21] but only up to bisimulation.

8 Conclusive remarks

This paper proposes a stochastic extension of CCS without replication [24]. With respect to other similar approaches, we propose an elegant structural operational semantics based on measure theory. For organizing the set of processes as a measurable space, we have chosen the σ-algebra defined by the structural congruence-closed sets of processes. This choice is motivated by practical modeling issues.

For defining the stochastic behavior, we introduce stochastic transition kernels. The definition, given for unspecified measurable spaces, generalizes rate transition systems [21, 8] and can be seen as a stochastic extension of Markov processes [5, 11, 10] and of Harsanyi type spaces [16, 27]. This fact presents stochastic bisimulation of STKs as particularly appropriate for logical characterization along the lines of Hennessy-Milner logics (applied to Markov processes [12, 7]) or along the lines of Aumann's system [1, 19, 33, 14] (applied to Harsanyi type spaces). Using a semantics on STKs, we define the stochastic bisimulation on processes and we prove that it is a congruence extending structural congruence.

The novelty of this work consists in the fact that the algebra of processes is axiomatized by structural congruence. This choice makes our stochastic process algebra appropriate for practical modeling purposes where various congruences can be relevant, as well as for extensions to other calculi with more complex equational axiomatizations such as π-calculus.

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Appendix

In this appendix we have collected the proofs of the major results and the detailed discussions of the examples presented in the paper.

Proof (Lemma 1). It is sufficient to show that for each $P \in \mathbb{P}$ and each $\alpha \in \mathbb{A}^+$,

$$\theta(P)(\alpha) : \Pi \to [0, \infty]$$

is a measure on the measurable space $(\mathbb{P}, \Pi)$. The proof follows the inductive steps of the construction in Definition 6.
Case 1: If $P = 0$, then for any $\alpha \in \mathbb{A}^+$, $\theta(0)(\alpha) = \omega$ which is the null measure.
Case 2: If $P = \alpha.Q$ for some $\alpha \in \mathbb{A}^+$, then $\theta(\alpha)(\alpha.Q)$ is a Dirac measure and for any $\alpha' \in \mathbb{A}^+ \setminus \{\alpha\}$, $\theta(\alpha.Q)(\alpha') = \omega$ which is the null measure.
Case 3: We show now that if $P = Q + R$, then $\theta(Q + R)(\alpha)$ is a measure.

First notice that for any $a \in \mathbb{A}$,

$$\theta(Q + R)(a)(\emptyset) = \theta(Q)(a)(\emptyset) + \theta(R)(a)(\emptyset) = 0 \text{ and}$$

$$\theta(Q + R)(\tau_{(r)})(\emptyset) = \sum_{r=s+t} [\theta(Q)(\tau_{(s)})(\emptyset) + \theta(R)(\tau_{(t)})(\emptyset)] = 0,$$

because, by the inductive hypothesis, $\theta(Q)(\alpha)$ and $\theta(R)(\alpha)$ are measures for any $\alpha \in \mathbb{A}^+$.

Consider now an arbitrary sequence of pairwise disjoint sets $(\mathcal{R}_i)_{i\in I} \in \Pi$. Then, for arbitrary $a \in \mathbb{A}$, $\theta(Q+R)(a)(\cup_{i\in I}\mathcal{R}_i) = \theta(Q)(a)(\cup_{i\in I}\mathcal{R}_i)+\theta(R)(a)(\cup_{i\in I}\mathcal{R}_i)$. But, from the inductive hypothesis, $\theta(Q)(a)(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I} \theta(Q)(a)(\mathcal{R}_i)$ and $\theta(R)(a)(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I} \theta(R)(a)(\mathcal{R}_i)$. Consequently,

$$\theta(Q + R)(a)(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I} \theta(Q)(a)(\mathcal{R}_i) + \sum_{i\in I} \theta(R)(a)(\mathcal{R}_i) =$$

$$= \sum_{i\in I} (\theta(Q)(a)(\mathcal{R}_i) + \theta(R)(a)(\mathcal{R}_i)) = \sum_{i\in I} \theta(Q + R)(a)(\mathcal{R}_i).$$

Similarly, for arbitrary $r \in \mathbb{Q}_+^\infty$,

$$\theta(Q + R)(\tau(r))(\cup_{i\in I}\mathcal{R}_i) = \sum_{r=s+t} [\theta(Q)(\tau_{(s)})(\cup_{i\in I}\mathcal{R}_i) + \theta(R)(\tau_{(t)})(\cup_{i\in I}\mathcal{R}_i)].$$

But, from the inductive hypothesis, $\theta(Q)(\tau_{(s)})(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I} \theta(Q)(\tau_{(s)})(\mathcal{R}_i)$ and $\theta(R)(\tau_{(t)})(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I} \theta(R)(\tau_{(t)})(\mathcal{R}_i)$. Consequently,

$$\theta(Q + R)(\tau_{(r)})(\cup_{i\in I}\mathcal{R}_i) = \sum_{r=s+t} [\sum_{i\in I} \theta(Q)(\tau_{(s)})(\mathcal{R}_i) + \sum_{i\in I} \theta(R)(\tau_{(t)})(\mathcal{R}_i)] =$$

$$= \sum_{i\in I} \sum_{r=s+t} [(\theta(Q)(\tau_{(s)})(\mathcal{R}_i) + \theta(R)(\tau_{(t)})(\mathcal{R}_i))] = \sum_{i\in I} \theta(Q + R)(\tau_{(r)})(\mathcal{R}_i).$$

Case 4: $P \equiv Q|R$.

Let $a \in \mathbb{A}$. $\theta(Q|R)(a)(\emptyset) = \theta(R)(a)(\emptyset_Q) + \theta(Q)(a)(\emptyset_R) = 0$, because $\emptyset_Q = \emptyset_R = \emptyset$ and, from the inductive hypothesis, $\theta(R)(a)$ and $\theta(Q)(a)$ are measures. Moreover, for arbitrary $r \in \mathbb{Q}_+^\infty$,

$$\theta(Q|R)(\tau_{(r)})(\emptyset) = \theta(R)(\tau_{(r)})(\emptyset_Q) + \theta(Q)(\tau_{(r)})(\emptyset_R)+$$

$$+\sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \emptyset}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \theta(Q)(a)(\mathcal{P}_1) \cdot \theta(R)(\overline{a})(\mathcal{P}_2)}{2}.$$

But $\emptyset_Q = \emptyset_R = \emptyset$ and $\mathcal{P}_1|\mathcal{P}_2 \subseteq \emptyset$ implies $\mathcal{P}_1 = \mathcal{P}_2 = \emptyset$. The inductive hypothesis guarantees that $\theta(R)(\tau_{(r)})(\emptyset) = \theta(Q)(\tau_{(s)})(\emptyset) = \theta(R)(\overline{a})(\emptyset) = \theta(Q)(a)(\emptyset) = 0$. Hence, $\theta(Q|R)(\tau_{(r)})(\emptyset) = 0$.

Consider now an arbitrary sequence of pairwise disjoint sets $(\mathcal{R}^i)_{i \in I} \in \Pi$ and let $\mathcal{P} = \cup_{i \in I} \mathcal{R}^i$. Then,

$$\theta(Q|R)(a)(\mathcal{P}) = \theta(R)(a)(\mathcal{P}_Q) + \theta(Q)(a)(\mathcal{P}_R).$$

Observe that $\mathcal{R}^i_Q$ and $\mathcal{R}^i_R$ are pairwise disjoint, because the sets $\mathcal{R}^i$ are pairwise disjoint. Consequently, using the inductive hypothesis, we obtain

$$\theta(Q|R)(a)(\mathcal{P}) = \sum_{i \in I} \theta(R)(a)(\mathcal{R}^i_Q) + \sum_{i \in I} \theta(Q)(a)(\mathcal{R}^i_R) =$$

$$= \sum_{i \in I} [\theta(R)(a)(\mathcal{R}^i_Q) + \theta(Q)(a)(\mathcal{R}^i_R)], \text{ i.e.,}$$

$$\theta(Q|R)(a)(\mathcal{P}) = \sum_{i \in I} \theta(Q|R)(a)(\mathcal{R}^i).$$

Consider an arbitrary $r \in \mathbb{Q}_+^\infty$.

$$\theta(Q|R)(\tau_{(r)})(\mathcal{P}) = \theta(R)(\tau_{(r)})(\mathcal{P}_Q) + \theta(Q)(\tau_{(r)})(\mathcal{P}_R)+$$

$$+\sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{P}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \theta(Q)(a)(\mathcal{P}_1) \cdot \theta(R)(\overline{a})(\mathcal{P}_2)}{2}.$$

As before,

$$\theta(Q|R)(\tau_{(r)})(\mathcal{P}) = \sum_{i \in I} \theta(R)(\tau_{(r)})(\mathcal{R}^i_Q) + \sum_{i \in I} \theta(Q)(\tau_{(r)})(\mathcal{R}^i_R)+$$

$$+\sum_{i \in I} \sum_{\mathcal{P}_1|\mathcal{P}_2 \in \mathcal{R}_i}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \theta(Q)(a)(\mathcal{P}_1) \cdot \theta(R)(\overline{a})(\mathcal{P}_2)}{2} =$$

$$= \sum_{i \in I} \theta(Q|R)(\tau_{(r)})(\mathcal{R}_i).$$

Proof (Lemma 2).

1. The proof goes similarly with the proof of Lemma 1.

Further, we only prove 3(b) and 3(c), the other cases being trivial.

3(b). Let $\mu = \mu' \;_{P}\!\otimes_{Q}\; \mu''$ and arbitrary $a \in \mathbb{A}$, $\mathcal{R} \in \Pi$. Then,

$$((\mu' \;_{P}\!\otimes_{Q}\; \mu'') \;_{P|Q}\!\otimes_{R}\; \mu''')(a)(\mathcal{R}) = (\mu \;_{P|Q}\!\otimes_{R}\; \mu''')(a)(\mathcal{R}) =$$

$$\mu(a)(\mathcal{R}_R) + \mu'''(a)(\mathcal{R}_{P|Q})$$

But $\mu(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}) \;_{P}\!\otimes_{Q}\; \mu''(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}_Q) + \mu''(a)(\mathcal{R}_P)$.

$$((\mu' \;_{P}\!\otimes_{Q}\; \mu'') \;_{P|Q}\!\otimes_{R}\; \mu''')(a)(\mathcal{R}) =$$

$$= (\mu'(a)((\mathcal{R}_R)_Q) + \mu''(a)((\mathcal{R}_R)_P)) + \mu'''(a)(\mathcal{R}_{P|Q}).$$

Observe that for arbitrary $P, Q \in \mathbb{P}$ and arbitrary $\mathcal{R} \in \Pi$, $(\mathcal{R}_P)_Q = \mathcal{R}_{P|Q}$. Using this, we obtain

$$((\mu' \;_{P}\!\otimes_{Q}\; \mu'') \;_{P|Q}\!\otimes_{R}\; \mu''')(a)(\mathcal{R}) =$$

$$= \mu'(a)(\mathcal{R}_{Q|R}) + \mu''(a)(\mathcal{R}_{P|R}) + \mu'''(a)(\mathcal{R}_{P|Q}).$$

In the same way we can prove that

$$\mu' \;_{P}\!\otimes_{Q|R}\; (\mu'' \;_{Q}\!\otimes_{R}\; \mu''')(a)(\mathcal{R}) =$$

$$= \mu'(a)(\mathcal{R}_{Q|R}) + \mu''(a)(\mathcal{R}_{P|R}) + \mu'''(a)(\mathcal{R}_{P|Q}).$$

Thus, $((\mu' \;_{P}\!\otimes_{Q}\; \mu'') \;_{P|Q}\!\otimes_{R}\; \mu''')(a)(\mathcal{R}) = (\mu' \;_{P}\!\otimes_{Q|R}\; (\mu'' \;_{Q}\!\otimes_{R}\; \mu'''))(a)(\mathcal{R})$.

We prove now that for arbitrary $r \in \mathbb{Q}_+^\infty$

$$((\mu' \;_{P}\!\otimes_{Q}\; \mu'') \;_{P|Q}\!\otimes_{R}\; \mu''')(\tau_{(r)})(\mathcal{R}) = (\mu' \;_{P}\!\otimes_{Q|R}\; (\mu'' \;_{Q}\!\otimes_{R}\; \mu'''))(\tau_{(r)})(\mathcal{R}).$$

As before, we have

$$((\mu' \;_{P}\!\otimes_{Q}\; \mu'') \;_{P|Q}\!\otimes_{R}\; \mu''')(\tau_{(r)})(\mathcal{R}) = (\mu \;_{P|Q}\!\otimes_{R}\; \mu''')(\tau_{(r)})(\mathcal{R}) =$$

$$= \mu(\tau_{(r)})(\mathcal{R}_R) + \mu'''(\tau_{(r)})(\mathcal{R}_{P|R}) + \sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{R}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu(a)(\mathcal{P}_1) \cdot \mu'''(\overline{a})(\mathcal{P}_2)}{2} =$$

$$= \mu'(\tau_{(r)})(\mathcal{R}_{R|Q}) + \mu''(\tau_{(r)})(\mathcal{R}_{R|P}) + \mu'''(\tau_{(r)})(\mathcal{R}_{P|R}) +$$

$$+ \sum_{\mathcal{Q}_1|\mathcal{Q}_2 \subseteq \mathcal{R}_R}^{b \in \mathbb{A}_r} \frac{\iota(b) \cdot \mu'(b)(\mathcal{Q}_1) \cdot \mu''(\bar{b})(\mathcal{Q}_2)}{2} + \sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{R}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu(a)(\mathcal{P}_1) \cdot \mu'''(\overline{a})(\mathcal{P}_2)}{2}.$$

But

$$\sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{R}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu(a)(\mathcal{P}_1) \cdot \mu'''(\overline{a})(\mathcal{P}_2)}{2} =$$

$$= \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot[\mu'(a)((\mathcal{P}_1)_Q)+\mu''(a)((\mathcal{P}_1)_P)]\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2} =$$

$$= \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)((\mathcal{P}_1)_Q)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2} + \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu''(a)((\mathcal{P}_1)_P)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2}.$$

Observe that, due to the way the sum is defined (and because pairing is an involution) we have that

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)((\mathcal{P}_1)_Q)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2} = \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_Q}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2}$$

and

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu''(a)((\mathcal{P}_1)_P)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2} = \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_P}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu''(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2}.$$

Consequently,

$$((\mu' \ {}_P{\otimes}_Q \ \mu'') \ {}_{P|Q}{\otimes}_R \ \mu''')(\tau_{(r)})(\mathcal{R}) =$$

$$= \mu'(\tau_{(r)})(\mathcal{R}_{R|Q}) + \mu''(\tau_{(r)})(\mathcal{R}_{R|P}) + \mu'''(\tau_{(r)})(\mathcal{R}_{P|R}) +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_R}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)(\mathcal{P}_1)\cdot\mu''(\overline{a})(\mathcal{P}_2)}{2} +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_Q}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2} + \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_P}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu''(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2}.$$

Similarly can be proved that

$$(\mu' \ {}_P{\otimes}_{Q|R} \ (\mu'' \ {}_Q{\otimes}_R \ \mu'''))(\tau)(\mathcal{R}) =$$

$$= \mu'(\tau_{(r)})(\mathcal{R}_{R|Q}) + \mu''(\tau_{(r)})(\mathcal{R}_{R|P}) + \mu'''(\tau_{(r)})(\mathcal{R}_{P|R}) +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_R}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)(\mathcal{P}_1)\cdot\mu''(\overline{a})(\mathcal{P}_2)}{2} +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_Q}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu'(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2} + \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_P}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu''(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2}.$$

3(c). We prove now that $\mu' \ {}_P{\otimes}_0 \ \omega^{\mathbb{A}^+} = \mu'$. Consider arbitrary $a \in \mathbb{A}$ and $\mathcal{R} \in \mathit{\Pi}$.

$$(\mu' \ {}_P{\otimes}_0 \ \omega^{\mathbb{A}^+})(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}_0) + \omega^{\mathbb{A}^+}(a)(R_P).$$

But $\mathcal{R}_0 = \mathcal{R}$ and $\omega^{\mathbb{A}^+}(a)(R_P) = 0$. Consequently,

$$(\mu' \;{}_P{\otimes}_0\; \omega^{\mathbb{A}^+})(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}).$$

We also have, for arbitrary $r \in \mathbb{Q}_+^\infty$,

$$(\mu' \;{}_P{\otimes}_0\; \omega^{\mathbb{A}^+})(\tau_{(r)})(\mathcal{R}) = \mu'(\tau_{(r)})(\mathcal{R}_0) + \omega^{\mathbb{A}^+}(\tau_{(r)})(R_P) +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{R}}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu'(a)(\mathcal{P}_1) \cdot \omega^{\mathbb{A}^+}(\overline{a})(\mathcal{P}_2)}{2}.$$

But $\omega^{\mathbb{A}^+}(\overline{a})(\mathcal{P}_1) = \omega^{\mathbb{A}^+}(\tau_{(r)})(\mathcal{R}_P) = 0$ and $\mathcal{R}_0 = \mathcal{R}$, where from we obtain

$$(\mu' \;{}_P{\otimes}_0\; \omega^{\mathbb{A}^+})(\tau_{(r)})(\mathcal{R}) = \mu'(\tau_{(r)})(\mathcal{R}).$$

Proof (Lemma 3). The proof is done by induction on the structures of P and Q following the axioms of the structural congruence.

The case $P = R'|S$, $Q = R''|S$ with $R' \equiv R''$.
Suppose that $S \to \mu'$ and $R' \to \mu''$ (from the inductive hypothesis, $R'' \to \mu''$). Then, $\mu = \mu'' \;{}_{R'}{\otimes}_S\; \mu'$. Using 4(a) of Lemma 2, we obtain that $\mu = \mu'' \;{}_{R''}{\otimes}_S\; \mu'$ and, by (Par), $Q \to \mu$.

The case $P = R' + S$, $Q = R'' + S$ with $R' \equiv R''$.
Suppose that $S \to \mu'$ and $R' \to \mu''$ (from the inductive hypothesis, $R'' \to \mu''$). Then, by (Sum), $Q \to \mu'' \oplus \mu'$. But $\mu = \mu'' \oplus \mu'$.

The case $P = \alpha.R$, $Q = \alpha.S$ with $R \equiv S$.
We have $\mu = \alpha_R$ and $S \to \alpha_S$. As $R \equiv S$, we obtain that $\mu = \alpha_S$, i.e., $Q \to \mu$.

The case $P = R|S$, $Q = S|R$.
Suppose that $R \to \mu'$ and $S \to \mu''$. Then $Q \to \mu'' \;{}_S{\otimes}_R\; \mu'$ and $\mu = \mu' \;{}_R{\otimes}_S\; \mu''$. But we proved in Lemma 2 that $\mu'' \;{}_S{\otimes}_R\; \mu' = \mu' \;{}_R{\otimes}_S\; \mu''$.

The case $P = (R|S)|T$, $Q = R|(S|T)$.
Suppose that $R \to \mu'$, $S \to \mu''$ and $T \to \mu'''$. Then $Q \to \mu' \;{}_R{\otimes}_{S|T}(\mu'' \;{}_S{\otimes}_T \mu''')$ and $\mu = (\mu' \;{}_R{\otimes}_S \mu'') \;{}_{R|S}{\otimes}_T \mu'''$. But we proved in Lemma 2 that $\mu' \;{}_R{\otimes}_{S|T}(\mu'' \;{}_S{\otimes}_T \mu''') = (\mu' \;{}_R{\otimes}_S\; \mu'') \;{}_{R|S}{\otimes}_T\; \mu'''$.

The case $Q = P|0$.
$Q \to \mu \;{}_P{\otimes}_0\; \omega^{\mathbb{A}^+}$. But, from Lemma 2, $\mu \;{}_P{\otimes}_0\; \omega^{\mathbb{A}^+} = \mu$.

The cases $[P = R + S$ and $Q = S + R]$, $[P = (R + S) + T$ and $Q = R + (S + T)]$ and $[Q = P + 0]$.
These are consequences of the fact that $(\Delta(\mathbb{P})^{\mathbb{A}^+}, \oplus, \omega^{\mathbb{A}^+})$ is a commutative monoid (Lemma 2).

Proof (Example 1). 1. Suppose that $a.P|a.P \to \mu$. We know that $a.P \to a_P$. Hence,

$$\mu = a_P \ {}_{a.P}\!\otimes_{a.P}\ a_P.$$

Then, $\mu(a)((a.P|P)^{\equiv}) = (a_P \ {}_{a.P}\!\otimes_{a.P}\ a_P)(a)((a.P|P)^{\equiv}) =$

$$a_P(a)((a.P|P)^{\equiv}_{a.P}) + a_P(a)((a.P|P)^{\equiv}_{a.P}).$$

But $(a.P|P)^{\equiv}_{a.P} = P^{\equiv}$, consequently,

$$\mu(a)((a.P|P)^{\equiv}) = 2 \cdot a_P(a)(P^{\equiv}) = 2 \cdot \iota(a) = 2r.$$

This proves that, indeed, $a.P|a.P \xrightarrow{a,2r} a.P|P$.

2. Suppose that $a.P|\overline{a}.Q \to \mu$. We know that $a.P \to a_P$ and $\overline{a}.Q \to \overline{a}_Q$. Hence,

$$\mu = a_P \ {}_{a.P}\!\otimes_{\overline{a}.Q}\ \overline{a}_Q.$$

Then, $\mu(\tau_{(r)})((P|Q)^{\equiv}) = (a_P \ {}_{a.P}\!\otimes_{\overline{a}.Q}\ \overline{a}_Q)(\tau_{(r)})((P|Q)^{\equiv}) =$

$$= a_P(\tau_{(r)})((P|Q)^{\equiv}_{a.P}) + \overline{a}_Q(\tau_{(r)})((P|Q)^{\equiv}_{\overline{a}.Q}) +$$

$$+ \sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq (P|Q)^{\equiv}}^{b \in \mathbb{A}_r} \frac{\iota(b) \cdot a_P(b)(\mathcal{P}_1) \cdot \overline{a}_Q(\overline{b})(\mathcal{P}_2)}{2}.$$

Observe that $a_P(\tau_{(r)})((P|Q)^{\equiv}_{a.P}) = \overline{a}_Q(\tau_{(r)})((P|Q)^{\equiv})_{\overline{a.Q}} = 0$ and

$$\sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq (P|Q)^{\equiv}}^{b \in \mathbb{A}_r} \frac{\iota(b) \cdot a_P(b)(\mathcal{P}_1) \cdot \overline{a}_Q(\overline{b})(\mathcal{P}_2)}{2} =$$

$$= \frac{\iota(a) \cdot a_P(a)(P^{\equiv}) \cdot \overline{a}_Q(\overline{a})(Q^{\equiv})}{2} + \frac{\iota(\overline{a}) \cdot \overline{a}_Q(\overline{a})(Q^{\equiv}) \cdot \overline{\overline{a}}_P(\overline{\overline{a}})(P^{\equiv})}{2} = \frac{2r}{2} = r.$$

And this proves that $a.P|\overline{a}.Q \xrightarrow{\tau_{(r)};r} P|Q$.

3.Suppose that $R = (a.P_1 + b.P_2)|(\overline{a}.Q_1 + c.Q_2) \to \mu$. We know that $a.P_1 \to a_{P_1}$, $b.P_2 \to b_{P_2}$, $\overline{a}.Q_1 \to \overline{a}_{Q_1}$ and $c.Q_2 \to c_{Q_2}$. Hence,

$$\mu = (a_{P_1} + b_{P_2}) \ {}_{(a.P_1+b.P_2)}\!\otimes_{(\overline{a}.Q_1+c.Q_2)}\ (\overline{a}_{Q_1} + c_{Q_2}).$$

We prove that $\mu(\tau_{(r)})((P_1|Q_1)^{\equiv}) = r$.

$$[(a_{P_1} + b_{P_2}) \ {}_{(a.P_1+b.P_2)}\!\otimes_{(\overline{a}.Q_1+c.Q_2)}\ (\overline{a}_{Q_1} + c_{Q_2})](\tau_{(r)})((P_1|Q_1)^{\equiv}) =$$

$$[a_{P_1} + b_{P_2}](\tau_{(r)})((P_1|Q_1)^{\equiv}_{(\overline{a}.Q_1+c.Q_2)}) + [\overline{a}_{Q_1} + c_{Q_{\overline{2}}^{\equiv}}](\tau_{(r)})((P_1|Q_1)^{\equiv}_{(a.P_1+b.P_2)}) +$$

$$+ \sum_{\mathcal{P}|\mathcal{Q} \subseteq (P_1|Q_1)^{\equiv}}^{d \in \mathbb{A}_r} \frac{\iota(d) \cdot [a_{P_1} + b_{P_2}](d)(\mathcal{P}) \cdot [\overline{a}_{Q_1} + c_{Q_2}](\overline{d})(\mathcal{Q})}{2}.$$

But

$[a_{P_1} + b_{P_2}](\tau_{(r)})((P_1|Q_1)^{\equiv}_{(\overline{a}.Q_1+c.Q_2)}) = [\overline{a}_{Q_1} + c_{Q_2}](\tau_{(r)})((P_1|Q_1)^{\equiv}_{(a.P_1+b.P_2)}) = 0$

and

$$\sum_{\mathcal{P}|\mathcal{Q}\subseteq(P_1|Q_1)^{\equiv}}^{d\in\mathbb{A}_r} \frac{\iota(d)\cdot[a_{P_1}+b_{P_2}](d)(\mathcal{P})\cdot[\overline{a}_{Q_1}+c_{Q_2}](\overline{d})(\mathcal{Q})}{2} =$$

$$= \frac{\iota(a)\cdot a_{P_1}(a)(P_1^{\equiv})\cdot\overline{a}_{Q_1}(\overline{a})(Q_1^{\equiv})}{2} + \frac{\iota(\overline{a})\cdot\overline{a}_{Q_1}(\overline{a})(Q_1^{\equiv})\cdot\overline{\overline{a}}_{P_1}(\overline{\overline{a}})(P_1^{\equiv})}{2} = \frac{2r}{2} = r.$$

Proof (Lemma 4). We prove it by induction on the syntactic structure of P.

The case $P = 0$: the two sets are empty.

The case $P = \tau_{(r)}.R$: $P \to \tau_{(r)}\ _R$, implying that $P \xrightarrow{\tau_{(r)},r} R$ and for any $\alpha \in \mathbb{A}^+ \setminus \{\tau_{(r)}\}$, $P \xrightarrow{\alpha,0} R$. Moreover, for any $Q \not\equiv R$, and any $\alpha \in \mathbb{A}^+$, $P \xrightarrow{\alpha,0} Q$.

The case $P = P_1 + P_2$: suppose that $P \to \mu$ and $P_i \to \mu_i$ for $i = 1, 2$. Then, for arbitrary $a \in \mathbb{A}$ and $Q \in \mathbb{P}$, $\mu(a)(Q^{\equiv}) = \mu_1(a)(Q^{\equiv}) + \mu_2(a)(Q^{\equiv})$. As there are a finite number of $a \in \mathbb{A}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu_1(a)(Q^{\equiv}) \neq 0$ and a finite number of $a \in \mathbb{A}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu_2(a)(Q^{\equiv}) \neq 0$, we deduce that there exist a finite number of $a \in \mathbb{A}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu(a)(Q^{\equiv}) \neq 0$. Similarly, for arbitrary $r \in \mathbb{Q}^{\infty}_{+}$, $\mu(\tau_{(r)})(Q^{\equiv}) = \sum_{r=s+t} \mu_1(\tau_{(s)})(Q^{\equiv}) + \mu_2(\tau_{(t)})(Q^{\equiv})$. There exist a finite number of $s \in \mathbb{Q}^{\infty}_{+}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu_1(\tau_{(s)})(Q^{\equiv}) \neq 0$ and there exist a finite number of $t \in \mathbb{Q}^{\infty}_{+}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu_2(\tau_{(t)})(Q^{\equiv}) \neq 0$. These imply that there exist a finite number of $r \in \mathbb{Q}^{\infty}_{+}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu(\tau_{(r)})(Q^{\equiv}) \neq 0$.

The case $P = P_1|P_2$: suppose that $P \to \mu$ and $P_i \to \mu_i$ for $i = 1, 2$. Then, for arbitrary $a \in \mathbb{A}$ and $Q \in \mathbb{P}$, $\mu(a)(Q^{\equiv}) = \mu_1(a)(Q^{\equiv}_{\overline{P_2}}) + \mu_2(a)(Q^{\equiv}_{\overline{P_1}})$. As there are a finite number of $a \in \mathbb{A}$ and a finite number of $R \in \mathbb{P}$ such that $\mu_1(a)(R^{\equiv}) \neq 0$ and a finite number of $a \in \mathbb{A}$ and a finite number of $R \in \mathbb{P}$ such that $\mu_2(a)(R^{\equiv}) \neq 0$, we deduce that there exist a finite number of $a \in \mathbb{A}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu(a)(Q^{\equiv}) \neq 0$. Similarly, for arbitrary $r \in \mathbb{Q}^{\infty}_{+}$,

$$\mu(\tau_{(r)})(Q^{\equiv}) = \mu_1(\tau_{(r)})(Q^{\equiv}_{\overline{P_2}}) + \mu_2(\tau_{(r)})(Q^{\equiv}_{\overline{P_1}}) +$$

$$+ \sum_{\mathcal{Q}_1|\mathcal{Q}_2\subseteq Q^{\equiv}}^{a\in\mathbb{A}_r} \frac{\iota(a)\cdot\mu_1(a)(\mathcal{Q}_1)\cdot\mu_2(\overline{a})(\mathcal{Q}_2)}{2}.$$

There exist a finite number of $s \in \mathbb{Q}^{\infty}_{+}$ and a finite number of $R \in \mathcal{P}$ such that $\mu_1(\tau_{(s)})(R^{\equiv}) \neq 0$, there exist a finite number of $a \in \mathbb{A}_r$ and a finite number of $\mathcal{R} \in \mathit{\Pi}$ such that $\mu_1(a)(\mathcal{R}) \neq 0$, there exist a finite number of $s \in \mathbb{Q}^{\infty}_{+}$ and a finite number of $R \in \mathbb{P}$ such that $\mu_2(\tau_{(s)})(R^{\equiv}) \neq 0$ and there exist a finite number of $\overline{a} \in \mathbb{A}_r$ and a finite number of $\mathcal{R} \in \mathit{\Pi}$ such that $\mu_2(\overline{a})(\mathcal{R}) \neq 0$. These imply that there exist a finite number of $r \in \mathbb{Q}^{\infty}_{+}$ and a finite number of $Q \in \mathbb{P}$ such that $\mu(\tau_{(r)})(Q^{\equiv}) \neq 0$.

Proof (Theorem 1). Before proceeding with the proof let's notice that if we have two equivalence relations $\mathcal{R}_1, \mathcal{R}_2$ on a set M, there exists an equivalence relation $\mathcal{R}$ on M such that $R_1 \cup \mathcal{R}_2 \subseteq \mathcal{R}$. Moreover, each R-equivalence class can be seen as the reunion of $\mathcal{R}_1$-equivalence classes as well as the reunion of $\mathcal{R}_2$-equivalence classes. The same result is true if we start from a denumerable set of equivalence relations.

We prove now that $\sim$ is an equivalence relation. Reflexivity and symmetry are trivial. We prove the transitivity.

Suppose that $P \sim Q$ and $Q \sim R$, $P \to \mu$, $Q \to \mu'$ and $R \to \mu''$. Then, there exist two stochastic bisimulation relations $\mathcal{R}_1, \mathcal{R}_2$ such that $(P, Q) \in \mathcal{R}_1$ and $(Q, R) \in \mathcal{R}_2$. Let $\mathcal{R}$ be the smallest equivalence relation such that $\mathcal{R}_1 \cup \mathcal{R}_2 \subseteq \mathcal{R}$. Consider arbitrary $\alpha \in \mathbb{A}^+$ and $C \in \Pi(\mathcal{R})$. Observe that, by definition, $\Pi(\mathcal{R}) = \Pi \cap \mathbb{P}^{\mathcal{R}}$, where we denoted by $\mathbb{P}^{\mathcal{R}}$ the set of $\mathcal{R}$-equivalence classes. Hence, $C \in \mathbb{P}^{\mathcal{R}}$ and because $C \in \Pi$ and Π is denumerable, we obtain that there exist $(C_1^i)_{i\in I} \subseteq \mathbb{P}^{\mathcal{R}_1}$ and $(C_2^j)_{j\in J} \subseteq \mathbb{P}^{\mathcal{R}_2}$ at most denumerable sets of $\mathcal{R}_1$ and respectively $\mathcal{R}_2$-equivalence classes, such that

$$C = \bigcup_{i\in I} C_1^i = \bigcup_{j\in J} C_2^j.$$

We also assume that the elements of $(C_1^i)_{i\in I}$ are pairwise distinct hence, (because they are equivalence classes) are pairwise disjoint. The same about $(C_2^j)_{j\in J}$.

Because $(P, Q) \in \mathcal{R}_1$, we have that for each $C_i \in \Pi(\mathcal{R}_1) = \Pi \cap \mathbb{P}^{\mathcal{R}_1}$ and each $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C_i) = \mu'(\alpha)(C_i)$.

Because $(Q, R) \in \mathcal{R}_2$, we have that for each $C_j \in \Pi(\mathcal{R}_2) = \Pi \cap \mathbb{P}^{\mathcal{R}_2}$ and each $\alpha \in \mathbb{A}^+$, $\mu'(\alpha)(C_j) = \mu''(\alpha)(C_j)$.

We show that for each $C \in \Pi(\mathcal{R})$ and each $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu''(\alpha)(C)$. Because $\mu(\alpha), \mu'(\alpha)$ and $\mu''(\alpha)$ are measures, we obtain

$$\mu(\alpha)(C) = \sum_{i\in I} \mu(\alpha)(C_i) = \sum_{i\in I} \mu'(\alpha)(C_i) = \mu'(\alpha)(C).$$

Similarly,

$$\mu'(\alpha)(C) = \sum_{j\in J} \mu'(\alpha)(C_j) = \sum_{j\in J} \mu''(\alpha)(C_j) = \mu''(\alpha)(C).$$

Hence, $\mu(\alpha)(C) = \mu''(\alpha)(C)$ proving that $\mathcal{R}$ is a stochastic bisimulation and concluding the transitivity proof.

For showing that $P \sim Q$ iff for any $C \in \Pi(\sim)$ and any $\alpha \in \mathbb{A}$, $\mu(\alpha)(C) = \mu'(\alpha)(C)$, we proceed as before, observing that $P \sim Q$ implies the existence of a bisimulation relation $\mathcal{R}$ such that $(P, Q) \in \mathcal{R}$. We can show that each $C \in \Pi(\sim)$ can be represented as a reunion of (at most denumerable) pairwise disjoint measurable $\mathcal{R}$-equivalence classes and, using the fact that $\mu(\alpha), \mu'(\alpha)$ are measures we can show that $\mu(\alpha)(C) = \mu'(\alpha)(C)$.

Proof (Example 2). 1. For $a, b \in \mathbb{A}$, $a \neq b$ we have $a.0 \not\sim b.0$, because $a.0 \rightarrow a_0 \in \Delta(\mathbb{P})^{\mathbb{A}^+}$ and $b.0 \rightarrow b_0 \in \Delta(\mathbb{P})^{\mathbb{A}^+}$ and, by definition, a_0 and b_0 agree on no $\alpha \in \mathbb{A}^+$. Concretely, if we take $C \in \mathbb{P}^{\sim}$ such that $0 \in C$, $a_0(a)(C) = 1$ while $b_0(a)(C) = 0$. Similarly, for $r \neq s$, $\tau_{(r)}.0 \not\sim \tau_{(s)}.0$ because $\tau_{(r)}.0 \rightarrow \tau_{(r)0}$ and $\tau_{(s)}.0 \rightarrow \tau_{(s)0}$ and, if we take $C \in \mathbb{P}^{\sim}$ such that $0 \in C$, $\tau_{(r)0}(\tau_{(r)})(C) = r$ while $\tau_{(s)0}(\tau_{(r)})(C) = 0$.

2. Observe that $a.0|b.0 \rightarrow [a_0\ {}_{a.0}\!\otimes_{b.0}\ b_0]$ and $a.b.0 + b.a.0 \rightarrow [a_{b.0} \oplus b_{a.0}]$. Now it is trivial to verify that, for arbitrary $C \in \mathbb{P}^{\sim}$, we have

$$[a_0\ {}_{a.0}\!\otimes_{b.0}\ b_0](x)(C) = a_0(x)(C_{b.0}) + b_0(a)(C_{b.0}) = \begin{cases} 1 & \text{,if } x = a, b.0 \in C \\ 0 & \text{,if } x = a, b.0 \notin C \\ 1 & \text{,if } x = b, a.0 \in C \\ 0 & \text{,if } x = b, a.0 \notin C \\ 0 & \text{,if } x \notin \{a, b\} \end{cases}$$

and

$$[a_{b.0} \oplus b_{a.0}](x)(C) = a_{b.0}(x)(C) + b_{a.0}(a)(C) = \begin{cases} 1 & \text{,if } x = a, b.0 \in C \\ 0 & \text{,if } x = a, b.0 \notin C \\ 1 & \text{,if } x = b, a.0 \in C \\ 0 & \text{,if } x = b, a.0 \notin C \\ 0 & \text{,if } x \notin \{a, b\} \end{cases}$$

which proves $a.0|b.0 \sim a.b.0 + b.a.0$. Similarly can be proved that for any $\alpha, \alpha' \in \mathbb{A}^+$ we have $\alpha.0|\alpha'.0 \sim \alpha.\alpha'.0 + \alpha'.\alpha.0$.

Proof (Theorem 2). Suppose that $P \rightarrow \mu$. $P \equiv Q$ implies (Lemma 3) that $Q \rightarrow \mu$. As for any $\sim$-equivalence class C and any $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu(\alpha)(C)$, we obtain $P \sim Q$.

Proof (Theorem 3).

For the proof of Theorem 3 we need the next lemma.

Lemma 6. *For arbitrary $P, Q, R \in \mathbb{P}$, if $P \sim Q$, then $P|R \sim Q|R$.*

Proof (Lemma 6). We prove the lemma inductively on the structure of the processes involved. For doing this, we first define the complexity of a process as the number of algebraic operators appearing in its syntax.

Let $cx : \mathbb{P} \rightarrow \mathbb{N}$ by
$cx(0) = 0$, $cx(\alpha.P) = cx(P) + 1$ and $cx(P|Q) = cx(P + Q) = cx(P) + cx(Q)$.

Observe that the complexity of a process is strictly related to the behavior of the process. Indeed, if for some $r \neq 0$, $P \xrightarrow{\alpha,r} Q$, then $cx(P) > cx(Q)$.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{N}^2$ we write $(x_1, x_2) < (y_1, y_2)$ iff for each $i = 1, 2$, $x_i \leq y_i$ and for some $j = 1, 2$, $x_j < y_j$. With this notation, we will prove the lemma inductively on $(max(cx(P), cx(Q)), cx(R))$.

The base case is trivial, so we prove, in what follows, the inductive step.

Suppose that for any $P', Q', R' \in \mathbb{P}$ with

$$(max(cx(P'), cx(Q')), cx(R')) < (max(cx(P), cx(Q)), cx(R))$$

we have that if $P' \sim Q'$, then $P'|R' \sim Q'|R'$. And we show that if $P \sim Q$, then $P|R \sim Q|R$.

Suppose that $P \rightarrow \mu$, $Q \rightarrow \eta$ and $R \rightarrow \rho$. Then, $P|R \rightarrow \mu \ _P\otimes_R \ \rho$ and $Q|R \rightarrow \eta \ _Q\otimes_R \ \rho$. For showing $P|R \sim Q|R$, it is sufficient to show that for arbitrary $\alpha \in \mathbb{A}^+$ and $C \in \mathbb{P}^\sim$,

$$(\mu \ _P\otimes_R \ \rho)(\alpha)(C) = (\eta \ _Q\otimes_R \ \rho)(\alpha)(C).$$

The case $\alpha = a \in \mathbb{A}$.
Due to Lemma 4 we can assume that:

- there exists a finite set of processes $\mathcal{P} = \{P_1^1, ..., P_1^{n_1}, ..., P_k^1, ..., P_k^{n_k}\}$, pairwise non structural congruent, such that $P \xrightarrow{a,0} \mathbb{P} \setminus \mathcal{P}$ and $P \xrightarrow{a,p_i^j} P_i^j$ for some $p_i^j \neq 0$; in addition, for each $i = 1..k$ and each $j, j' \in \{1, ..n_i\}$, $P_i^j \sim P_i^{j'}$ and for $i \neq i'$, $x = 1..n_i$, $x' = 1..n_{i'}$, $P_i^x \not\sim P_{i'}^{x'}$; let $p_i = \sum_{j=1..n_i} p_i^j$;
- there exists a finite set of processes $\mathcal{Q} = \{Q_1^1, ..., Q_1^{m_1}, ..., Q_l^1, ..., Q_l^{m_l}\}$, pairwise non structural congruent, such that $Q \xrightarrow{a,0} \mathbb{P} \setminus \mathcal{Q}$ and $Q \xrightarrow{a,q_i^j} Q_i^j$ for some $q_i^j \neq 0$; in addition, for each $i = 1..l$ and each $j, j' \in \{1, ..m_i\}$, $Q_i^j \sim Q_i^{j'}$ and for $i \neq i'$, $x = 1..m_i$, $x' = 1..m_{i'}$, $Q_i^x \not\sim Q_{i'}^{x'}$; let $q_i = \sum_{j=1..m_i} q_i^j$;
- there exists a finite set of processes $\mathcal{R} = \{R_1^1, ..., R_1^{u_1}, ..., R_v^1, ..., R_v^{u_v}\}$, pairwise non structural congruent, such that $R \xrightarrow{a,0} \mathbb{P} \setminus \mathcal{R}$ and $R \xrightarrow{a,r_i^j} R_i^j$ for some $r_i^j \neq 0$; in addition, for each $i = 1..v$ and each $j, j' \in \{1, ..u_i\}$, $R_i^j \sim R_i^{j'}$ and for $i \neq i'$, $x = 1..u_i$, $x' = 1..u_{i'}$, $R_i^x \not\sim r_{i'}^{x'}$; let $r_i = \sum_{j=1..u_i} r_i^j$;

Observe that $P \sim Q$ implies $k = l$, we can suppose that $P_i^j \sim Q_i^{j'}$ and for each $i = 1..k$, $p_i = q_i$.

For arbitrary $C \in \mathbb{P}^\sim$,

$$(\mu \ _P\otimes_R \rho)(a)(C) = \mu(a)(C_R) + \rho(a)(C_P) = \sum_{(P_1|R)^\equiv \subseteq C} \mu(a)(P_1^\equiv) + \sum_{(R_1|P)^\equiv \subseteq C} \rho(a)(R_1^\equiv),$$

and

$$(\eta \ _Q\otimes_R \rho)(a)(C) = \eta(a)(C_R) + \rho(a)(C_Q) = \sum_{(Q_1|R)^\equiv \subseteq C} \eta(a)(Q_1^\equiv) + \sum_{(R_1|Q)^\equiv \subseteq C} \rho(a)(R_1^\equiv).$$

If there exists $i_1, ..i_t$ such that for each $i \in \{i_1, ..i_t\}$ and only for them there exists $j \in \{1..n_i\}$ with $P_i^j|R \in C$, then, from the inductive hypothesis we have that for each $j' = 1..n_i$, $P_i^{j'}|R \in C$. Moreover, if $P'|R \in C$ such that $P|R \xrightarrow{a,s} P'|R$ for $s \neq 0$, then there exist i, j such that $P' \equiv P_i^j$. Consequently,

$$\sum_{(P_1|R)^{\equiv} \subseteq C} \mu(a)(P_1^{\equiv}) = \sum_{s=1..t} p_s.$$

But $P_i^j \sim Q_i^{j'}$ where from, using the inductive hypothesis, $Q_i^{j'}|R \in C$. Further, a similar argument as before gives

$$\sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(a)(Q_1^{\equiv}) = \sum_{s=1..t} p_s.$$

On the other hand, if there exists no i and j such that $P_i^j|R \in C$, from $P \sim Q$ we can prove that there is no i, j such that $Q_i^j|R \in C$, where from we obtain

$$\sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(a)(Q_1^{\equiv}) = \sum_{(P_1|R)^{\equiv} \subseteq C} \mu(a)(P_1^{\equiv}) = 0.$$

Observe now that $P \sim Q$ implies, using the inductive hypothesis, that $P|R_i^j \sim Q|R_i^j$, i.e., $R_i^j|P \in C$ iff $Q|R_i^j \in C$. Moreover, if $P|R_i^j \in C$, $P|R_i^{j'} \in C$ for any $j' = 1, ..u_i$ and for any $i' \neq i$, $P|R_{i'}^{j''} \notin C$. Hence, supposing that $R_i^j|P \in C$, we obtain

$$\sum_{(R'|P)^{\equiv} \subseteq C} \rho(a)(R') = \sum_{(R'|Q)^{\equiv} \subseteq C} \rho(a)(R') = r_i.$$

Else, if for no i, j, $R_i^j|P \in C$ we also have that for no i, j $R_i^j|Q \in C$ implying

$$\sum_{(R'|P)^{\equiv} \subseteq C} \rho(a)(R') = \sum_{(R'|Q)^{\equiv} \subseteq C} \rho(a)(R') = 0.$$

The case $\alpha = \tau_{(r)}$.
Due to Lemma 4 we can assume that:

- there exists a finite set of processes $\mathcal{P} = \{P_0^1, ..., P_0^{n_0}\}$, pairwise non structural congruent, such that $P \xrightarrow{\tau_{(r)}, p_0^j} P_0^j$ for some $p_0^j \neq 0$ and $P \xrightarrow{\tau_{(r)}, 0} \mathbb{P} \setminus \mathcal{P}$; in addition, there exists a finite set of actions $a \in \mathbb{A}_r$ with $P \xrightarrow{a,s} \mathbb{P}$ for some $s \neq 0$ and for each such a there exists a set $\{P_1^1, ..., P_1^{n_1}, ..., P_k^1, ..., P_k^{n_k}\}$ of processes, pairwise non structural congruent, such that $P \xrightarrow{a, p_i^j} P_i^j$ for some $p_i^j \neq 0$; moreover, for each $i = 0..k$ and $j, j' \in \{1, ..n_i\}$, $P_i^j \sim P_i^{j'}$ and for $i \neq i'$, $x = 1..n_i$, $x' = 1..n_{i'}$, $P_i^x \not\sim P_{i'}^{x'}$; let $p_i = \sum_{j=1..n_i} p_i^j$ for each $i = 0, ..k$;
- there exists a finite set of processes $\mathcal{Q} = \{Q_0^1, ..., Q_0^{m_0}\}$, pairwise non structural congruent, such that $Q \xrightarrow{\tau_{(r)}, q_0^j} Q_0^j$ for some $q_0^j \neq 0$ and $Q \xrightarrow{\tau_{(r)}, 0} \mathbb{P} \setminus \mathcal{Q}$; in addition, there exists a finite set of actions $a \in \mathbb{A}_r$ with $Q \xrightarrow{a,s} \mathbb{P}$ for some

$s \neq 0$ and for each such a there exists a set $\{Q_1^1, ..., Q_1^{m_1}, ..., P_l^1, ..., P_l^{m_l}\}$ of processes, pairwise non structural congruent, such that $Q \xrightarrow{a, q_i^j} Q_i^j$ for some $q_i^j \neq 0$; moreover, for each $i = 0..l$ and $j, j' \in \{1, ..m_i\}$, $Q_i^j \sim Q_i^{j'}$ and for $i \neq i'$, $x = 1..m_i$, $x' = 1..m_{i'}$, $Q_i^x \not\sim Q_{i'}^{x'}$; let $q_i = \sum_{j=1..n_i} q_i^j$ for each $i = 0, ..l$;

- there exists a finite set of processes $\mathcal{R} = \{R_0^1, ..., R_0^{u_0}\}$, pairwise non structural congruent, such that $R \xrightarrow{\tau_{(r)}, r_0^j} R_0^j$ for some $r_0^j \neq 0$ and $R \xrightarrow{\tau_{(r)}, 0} \mathbb{P} \setminus \mathcal{R}$; in addition, there exists a finite set of actions $a \in \mathbb{A}_r$ with $R \xrightarrow{a,s} \mathbb{P}$ for some $s \neq 0$ and for each such a there exists a set $\{R_1^1, ..., R_1^{n_1}, ..., R_k^1, ..., R_k^{n_k}\}$ of processes, pairwise non structural congruent, such that $R \xrightarrow{a, r_i^j} R_i^j$ for some $r_i^j \neq 0$; moreover, for each $i = 0..v$ and $j, j' \in \{1, ..u_i\}$, $R_i^j \sim R_i^{j'}$ and for $i \neq i'$, $x = 1..u_i$, $x' = 1..u_{i'}$, $R_i^x \not\sim R_{i'}^{x'}$; let $r_i = \sum_{j=1..n_i} p_i^j$ for each $i = 0, ..v$;

Observe that $P \sim Q$ implies, for each a having the mentioned properties, that $k = l$; we can suppose, without loosing generality, that $P_i^j \sim Q_i^{j'}$ and for each $i = 0..k$, $p_i = q_i$.

For arbitrary $C \in \mathbb{P}^{\sim}$,

$$(\mu \ {}_P\!\otimes_R \ \rho)(\tau_{(r)})(C) = \mu(\tau_{(r)})(C_R) + \rho(\tau_{(r)})(C_P) +$$

$$+ \sum_{(P_1|P_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu(a)(P_1^{\equiv}) \cdot \rho(\overline{a})(P_2^{\equiv})}{2} =$$

$$= \sum_{(P_1|R)^{\equiv} \subseteq C} \mu(\tau_{(r)})(P_1^{\equiv}) + \sum_{(R_1|P)^{\equiv} \subseteq C} \rho(\tau_{(r)})(R_1^{\equiv}) +$$

$$+ \sum_{(P_1|P_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \mu(a)(P_1^{\equiv}) \cdot \rho(\overline{a})(P_2^{\equiv})}{2},$$

and

$$(\eta \ {}_Q\!\otimes_R \ \rho)(a)(C) = \eta(\tau_{(r)})(C_R) + \rho(\tau_{(r)})(C_Q) +$$

$$+ \sum_{(Q_1|Q_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \eta(a)(Q_1^{\equiv}) \cdot \rho(\overline{a})(Q_2^{\equiv})}{2} =$$

$$= \sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(\tau_{(r)})(Q_1^{\equiv}) + \sum_{(R_1|Q)^{\equiv} \subseteq C} \rho(\tau_{(r)})(R_1^{\equiv}) +$$

$$+ \sum_{(Q_1|Q_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}_r} \frac{\iota(a) \cdot \eta(a)(Q_1^{\equiv}) \cdot \rho(\overline{a})(Q_2^{\equiv})}{2}.$$

At this level we can demonstrate, using the same strategy as in the case $\alpha = a$, that

–

$$\sum_{(P_1|R)^{\equiv}\subseteq C} \mu(\tau_{(r)})(P_1^{\equiv}) = \sum_{(Q_1|R)^{\equiv}\subseteq C} \eta(\tau_{(r)})(Q_1 \equiv) = \sum_{i=1..t} p_i,$$

where $i_1, ..i_t$ are such that for each $i \in \{i_1, ..i_t\}$, there exists some j such that (hence, for all j) $P_i^j|R \in C$ and, from the inductive hypothesis, there exists j' such that (hence, for all j') $Q_i^{j'}|R \in C$;

– because $P|R_i^j \sim Q|R_i^j$,

$$\sum_{(R_1|P)^{\equiv}\subseteq C} \rho(\tau_{(r)})(R_1^{\equiv}) = \sum_{(R_1|Q)^{\equiv}\subseteq C} \rho(\tau_{(r)})(R_1^{\equiv}) = r_i,$$

where i is (the unique index) such that for some (hence, for all) j, j', $P|R_i^j, Q|R_i^{j'} \in C$.

– for each a as before we also have

$$\sum_{(P_1|P_2)^{\equiv}\subseteq C} \frac{\iota(a)\cdot\eta(a)(P_1^{\equiv})\cdot\rho(\overline{a})(P_2^{\equiv})}{2} = \sum_{(i,j)\in I} p_i \cdot q_j =$$

$$= \sum_{(P_1|P_2)^{\equiv}\subseteq C} \frac{\iota(a)\cdot\eta(a)(P_1^{\equiv})\cdot\rho(\overline{a})(P_2^{\equiv})}{2},$$

where I is the set of couples of indexes (i, j) such that for some x, y (hence, for all), $P_i^x|Q_j^y \in C$.

We can proceed now with the proof of Theorem 3.

1. If $P \sim P'$, then $\alpha.P \sim \alpha.P'$.
We have $\alpha.P \to \alpha_P$ and $\alpha.P' \to \alpha_{P'}$.

$$\alpha_P(x)(C) = \begin{cases} 1 & ,\text{if } x = \alpha \in \mathbb{A}, P \in C \\ \iota(\alpha) & ,\text{if } x = \alpha \notin \mathbb{A}, P \in C \\ 0 & ,\text{if } x = \alpha, P \notin C \\ 0 & ,\text{if } x = \alpha' \neq \alpha \end{cases}$$

and

$$\alpha_{P'}(x)(C) = \begin{cases} 1 & ,\text{if } x = \alpha \in \mathbb{A}, P' \in C \\ \iota(\alpha) & ,\text{if } x = \alpha \notin \mathbb{A}, P' \in C \\ 0 & ,\text{if } x = \alpha, P' \notin C \\ 0 & ,\text{if } x = \alpha' \neq \alpha \end{cases}$$

Because $P \sim P'$, we have that for any $C \in \mathbb{P}^{\sim}$, $P \in C$ iff $P' \in C$. From here, we derive that $\alpha_P(\alpha)(C) = \alpha_{P'}(\alpha)(C)$, i.e. $\alpha.P \sim \alpha.P'$.

2. If $P \sim P'$ and $Q \sim Q'$, then $P + Q \sim P' + Q'$.
Suppose that $P \to \mu$, $P' \to \mu'$, $Q \to \eta$ and $Q' \to \eta'$. Consider an arbitrary $C \in \mathbb{P}^{\sim}$.

For $a \in \mathbb{A}$, $(\mu \oplus \eta)(a)(C) = \mu(a)(C) + \eta(a)(C)$. But $P \sim P'$ and $Q \sim Q'$, i.e. for any $C \in \mathbb{P}^{\sim}$, $\mu(a)(C) = \mu'(a)(C)$ and $\eta(a)(C) = \eta'(a)(C)$. Hence, $\mu(a)(C) + \eta(a)(C) = \mu'(a)(C) + \eta'(a)(C) = (\mu' \oplus \eta')(a)(C)$.

We also have

$$(\mu \oplus \eta)(\tau_{(r)})(C) = \sum_{r=s+t} [\mu(\tau_{(s)})(C) + \eta(\tau_{(t)})(C)].$$

But $P \sim P'$ and $Q \sim Q'$, i.e. for any $C \in \mathbb{P}^{\sim}$, $\mu(\tau_{(s)})(C) = \mu'(\tau_{(s)})(C)$ and $\eta(\tau_{(t)})(C) = \eta'(\tau_{(t)})(C)$. Hence,

$$\sum_{r=s+t} [\mu(\tau_{(s)})(C)+\eta(\tau_{(t)})(C)] = \sum_{r=s+t} [\mu'(\tau_{(s)})(C)+\eta'(\tau_{(t)})(C)] = (\mu' \oplus \eta')(\tau_{(r)})(C).$$

3. If $P \sim P'$ and $Q \sim Q'$, then $P|Q \sim P'|Q'$.
Using Lemma 6, we obtain that $P \sim P'$ implies $P|P' \sim Q|P'$ and $Q \sim Q'$ implies $Q|P' \sim Q'|P'$. Further, the transitivity of $\sim$ proves $P|Q \sim P'|Q'$.