

$F_{W<:}$

Syntax

$K, L ::=$		Kinds
$Ty \mid \Pi(X<:A::K)L$		
$A, B ::=$		Constructors
$X \mid \text{Top}(K) \mid \lambda(X<:A::K)B \mid B(A) \mid A \rightarrow B \mid \forall(X<:A::K)B \mid \mu(X)A$		
$a, b ::=$		Values
$x \mid \lambda(x:A)b \mid b(a) \mid \lambda(X<:A::K)b \mid b(A') \mid \text{fold}(\mu(X)B\{X\}, a) \mid \text{unfold}(a) \mid \mu(x:A)b$		

Judgments

$E \vdash \diamond$	E is an environment
$E \vdash K \text{ kind}$	K is a kind
$E \vdash K <: L \text{ kind}$	K is a subkind of L (optional)
$E \vdash K \Leftrightarrow L \text{ kind}$	K and L are equivalent kinds
$E \vdash A :: K$	constructor A has kind K
$E \vdash A <: B :: K$	A is a subconstructor of B , both of kind K
$E \vdash A \Leftrightarrow B :: K$	A and B are equivalent constructors of kind K
$E \vdash a : A$	value a has type A
$E \vdash a \Leftrightarrow b : A$	a and b are equivalent values of type A

Abbreviations

Top	for	$\text{Top}(Ty)$	
$X :: K$	for	$X <: \text{Top}(K) :: K$	(in environments and binders)
$X <: A$	for	$X <: A :: Ty$	(in environments and binders)
X	for	$X <: \text{Top} :: Ty$	(in environments and binders)
$E \vdash A$	for	$E \vdash A :: Ty$	
$E \vdash A :: K$	for	$E \vdash A <: \text{Top}(K) :: K$	
$E \vdash A <: B$	for	$E \vdash A <: B :: Ty$	
$E \vdash A \Leftrightarrow B$	for	$E \vdash A \Leftrightarrow B :: Ty$	

Environment Formation ($E \vdash \diamond$)

$$\begin{array}{c} \text{(Env } \emptyset) \\ \hline \emptyset \vdash \diamond \end{array} \quad \begin{array}{c} \text{(Env X)} \\ \frac{E \vdash A :: K \quad X \notin \text{dom}(E)}{E, X <: A :: K \vdash \diamond} \end{array} \quad \begin{array}{c} \text{(Env x)} \\ \frac{E \vdash A \quad x \notin \text{dom}(E)}{E, x:A \vdash \diamond} \end{array}$$

Kind Formation ($E \vdash K \text{ kind}$)

$$\begin{array}{c} \text{(Kind Ty)} \\ \frac{E \vdash \diamond}{E \vdash \text{Ty kind}} \end{array} \quad \begin{array}{c} \text{(Kind } \Pi) \\ \frac{E, X <: A :: K \vdash L \text{ kind}}{E \vdash \Pi(X <: A :: K)L \text{ kind}} \end{array}$$

Kind Inclusion ($E \vdash K <: L \text{ kind}$) [optional]

$$\begin{array}{c} \text{(Kind Sub Refl)} \\ \frac{E \vdash K \leftrightarrow L \text{ kind}}{E \vdash K <: L \text{ kind}} \end{array} \quad \begin{array}{c} \text{(Kind Sub Trans)} \\ \frac{E \vdash K <: L \text{ kind} \quad E \vdash L <: M \text{ kind}}{E \vdash K <: M \text{ kind}} \end{array}$$

$$\begin{array}{c} \text{(Kind Sub Ty)} \\ \frac{E \vdash \diamond}{E \vdash \text{Ty} <: \text{Ty kind}} \end{array} \quad \begin{array}{c} \text{(Kind Sub } \Pi) \\ \frac{E \vdash A' <: A :: K \quad E, X <: A' :: K \vdash L <: L' \text{ kind}}{E \vdash \Pi(X <: A :: K)L <: \Pi(X <: A' :: K)L' \text{ kind}} \end{array}$$

Kind Equivalence ($E \vdash K \leftrightarrow L \text{ kind}$)

$$\begin{array}{c} \text{(Kind Eq Refl)} \\ \frac{E \vdash K \leftrightarrow L \text{ kind}}{E \vdash L \leftrightarrow K \text{ kind}} \end{array} \quad \begin{array}{c} \text{(Kind Eq Trans)} \\ \frac{E \vdash K \leftrightarrow L \text{ kind} \quad E \vdash L \leftrightarrow M \text{ kind}}{E \vdash K \leftrightarrow M \text{ kind}} \end{array}$$

$$\begin{array}{c} \text{(Kind Eq Ty)} \\ \frac{E \vdash \diamond}{E \vdash \text{Ty} \leftrightarrow \text{Ty kind}} \end{array} \quad \begin{array}{c} \text{(Kind Eq } \Pi) \\ \frac{E \vdash K \leftrightarrow K' \text{ kind} \quad E \vdash A \leftrightarrow A' :: K \quad E, X <: A :: K \vdash L \leftrightarrow L' \text{ kind}}{E \vdash \Pi(X <: A :: K)L \leftrightarrow \Pi(X <: A' :: K)L' \text{ kind}} \end{array}$$

Constructor Formation ($E \vdash A :: K$)

$$\begin{array}{c} \text{(Con Extension)} \\ \frac{E \vdash A :: K \quad E \vdash K \leftrightarrow L \text{ kind}}{E \vdash A :: L} \end{array} \quad \begin{array}{c} \text{(Con Subsumption) [Replaces (Con Extension), with kind inclusion]} \\ \frac{E \vdash A :: K \quad E \vdash K <: L \text{ kind}}{E \vdash A :: L} \end{array}$$

$$\begin{array}{c} \text{(Con X)} \\ \frac{E', X <: A :: K, E'' \vdash \diamond}{E', X <: A :: K, E'' \vdash X :: K} \end{array} \quad \begin{array}{c} \text{(Con Top)} \\ \frac{E \vdash K \text{ kind}}{E \vdash \text{Top}(K) :: K} \end{array}$$

$$\begin{array}{c} \text{(Con Abs)} \\ \frac{E, X <: A :: K \vdash B :: L}{E \vdash \lambda(X <: A :: K)B :: \Pi(X <: A :: K)L} \end{array} \quad \begin{array}{c} \text{(Con Appl)} \\ \frac{E \vdash B :: \Pi(X <: A :: K)L\{X\} \quad E \vdash A' <: A :: K}{E \vdash B(A') :: L\{A'\}} \end{array}$$

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$$\begin{array}{ccc}
 \text{(Type Arrow)} & \text{(Type All)} & \text{(Type Rec)} \\
 \frac{E \vdash A \quad E \vdash B}{E \vdash A \rightarrow B} & \frac{E, X <: A :: K \vdash B}{E \vdash \forall (X <: A :: K) B} & \frac{E, X \vdash A \quad A > X}{E \vdash \mu(X) A}
 \end{array}$$

Constructor Inclusion ($E \vdash A <: B :: K$)

$$\begin{array}{cc}
 \text{(Con Sub Refl)} & \text{(Con Sub Trans)} \\
 \frac{E \vdash A \leftrightarrow B :: K}{E \vdash A <: B :: K} & \frac{E \vdash A <: B :: K \quad E \vdash B <: C :: K}{E \vdash A <: C :: K}
 \end{array}$$

$$\begin{array}{cc}
 \text{(Con Sub X)} & \text{(Con Sub Top)} \\
 \frac{E', X <: A :: K, E'' \vdash \diamond}{E', X <: A :: K, E'' \vdash X <: A :: K} & \frac{E \vdash A :: K}{E \vdash A <: \text{Top}(K) :: K}
 \end{array}$$

$$\begin{array}{c}
 \text{(Con Sub Abs)} \\
 \frac{E, X <: A :: K \vdash B <: B' :: L}{E \vdash \lambda(X <: A :: K) B <: \lambda(X <: A :: K) B' :: \Pi(X <: A :: K) L}
 \end{array}$$

$$\begin{array}{c}
 \text{(Con Sub Appl)} \\
 \frac{E \vdash B <: B' :: \Pi(X <: A :: K) L \{X\} \quad E \vdash A' <: A :: K}{E \vdash B(A') <: B'(A') :: L \{A'\}}
 \end{array}$$

$$\begin{array}{cc}
 \text{(Sub Arrow)} & \text{(Sub All)} \\
 \frac{E \vdash A' <: A \quad E \vdash B <: B'}{E \vdash A \rightarrow B <: A' \rightarrow B'} & \frac{E \vdash A' <: A :: K \quad E, X <: A' :: K \vdash B <: B'}{E \vdash \forall (X <: A :: K) B <: \forall (X <: A' :: K) B'}
 \end{array}$$

$$\begin{array}{c}
 \text{(Sub Rec)} \\
 \frac{E \vdash \mu(X) A \quad E \vdash \mu(Y) B \quad E, Y, X <: Y \vdash A <: B}{E \vdash \mu(X) A <: \mu(Y) B}
 \end{array}$$

Constructor Equivalence ($E \vdash A \leftrightarrow B :: K$)

$$\begin{array}{cc}
 \text{(Con Eq Symm)} & \text{(Con Eq Trans)} \\
 \frac{E \vdash A \leftrightarrow B :: K}{E \vdash B \leftrightarrow A :: K} & \frac{E \vdash A \leftrightarrow B :: K \quad E \vdash B \leftrightarrow C :: K}{E \vdash A \leftrightarrow C :: K}
 \end{array}$$

$$\begin{array}{cc}
 \text{(Con Eq X)} & \text{(Con Eq Top)} \\
 \frac{E \vdash X :: K}{E \vdash X \leftrightarrow X :: K} & \frac{E \vdash K \text{ kind}}{E \vdash \text{Top}(K) \leftrightarrow \text{Top}(K) :: K}
 \end{array}$$

$$\begin{array}{c}
 \text{(Con Eq Abs)} \\
 \frac{E \vdash K \leftrightarrow K' \text{ kind} \quad E \vdash A \leftrightarrow A' :: K \quad E, X <: A :: K \vdash B \leftrightarrow B' :: L}{E \vdash \lambda(X <: A :: K) B \leftrightarrow \lambda(X <: A' :: K') B' :: \Pi(X <: A :: K) L}
 \end{array}$$

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(Con Eq Appl)

$$\frac{E \vdash B \leftrightarrow B' :: \Pi(X <: A'' :: K)L\{X\} \quad E \vdash A <: A'' :: K \quad E \vdash A \leftrightarrow A' :: K}{E \vdash B(A) \leftrightarrow B'(A') :: L\{A\}}$$

(Type Eq All)

$$\frac{E \vdash A \leftrightarrow A' :: K \quad E, X <: A :: K \vdash B \leftrightarrow B'}{E \vdash \forall(X <: A :: K)B \leftrightarrow \forall(X <: A' :: K)B'}$$

(Type Eq Arrow)

$$\frac{E \vdash A \leftrightarrow A' \quad E \vdash B \leftrightarrow B'}{E \vdash A \rightarrow B \leftrightarrow A' \rightarrow B' \text{ type}}$$

(Type Eq Rec)

$$\frac{E, X \vdash B \leftrightarrow B'}{E \vdash \mu(X)B \leftrightarrow \mu(X)B'}$$

(Con Eq Eta)

$$\frac{E \vdash B :: \Pi(X <: A :: K)L \quad X \notin \text{dom}(E)}{E \vdash (\lambda(X <: A :: K)B(X)) \leftrightarrow B :: \Pi(X <: A :: K)L}$$

(Con Eq Beta)

$$\frac{E \vdash (\lambda(X <: A :: K)B\{X\})(A') :: L \quad E \vdash A' <: A :: K}{E \vdash (\lambda(X <: A :: K)B)(A) \leftrightarrow B\{A'\} :: L}$$

Value Formation ($E \vdash a : A$)

(Val Subsumption)

$$\frac{E \vdash a : A \quad E \vdash A <: B}{E \vdash a : B}$$

(Val x)

$$\frac{E', x : A, E'' \vdash \diamond}{E', x : A, E'' \vdash x : A}$$

(Val Abs)

$$\frac{E, x : A \vdash b : B}{E \vdash \lambda(x : A)b : A \rightarrow B}$$

(Val Appl)

$$\frac{E \vdash b : A \rightarrow B \quad E \vdash a : A}{E \vdash b(a) : B}$$

(Val Abs2)

$$\frac{E, X <: A :: K \vdash b : B}{E \vdash \lambda(X <: A :: K)b : \forall(X <: A :: K)B}$$

(Val Appl2)

$$\frac{E \vdash b : \forall(X <: A :: K)B\{X\} \quad E \vdash A' <: A :: K}{E \vdash b(A') : B\{A'\}}$$

(Val Fold)

$$\frac{E \vdash a : B\{\mu(X)B\{X\}\}}{E \vdash \text{fold}(\mu(X)B\{X\}, a) : \mu(X)B\{X\}}$$

(Val Unfold)

$$\frac{E \vdash a : \mu(X)B\{X\}}{E \vdash \text{unfold}(a) : B\{\mu(X)B\{X\}\}}$$

(Val Rec)

$$\frac{E, x : A \vdash b : A}{E \vdash \mu(x : A)b : A}$$

Value equivalence ($E \vdash a \leftrightarrow b : A$)

(Val Eq Refl)

$$\frac{E \vdash a : A}{E \vdash a \leftrightarrow a : A}$$

(Val Eq Symm)

$$\frac{E \vdash a \leftrightarrow b : A}{E \vdash b \leftrightarrow a : A}$$

(Val Eq Trans)

$$\frac{E \vdash a \leftrightarrow b : A \quad E \vdash b \leftrightarrow c : A}{E \vdash a \leftrightarrow c : A}$$

$$F_{W<:}$$

$$\begin{array}{c}
\text{(Val Eq x)} \qquad \qquad \text{(Val Eq Top)} \\
\frac{E \vdash x : A}{E \vdash x \leftrightarrow x : A} \qquad \frac{E \vdash a : \text{Top} \quad E \vdash b : \text{Top}}{E \vdash a \leftrightarrow b : \text{Top}} \\
\\
\text{(Val Eq Abs)} \qquad \qquad \qquad \text{(Val Eq Appl)} \\
\frac{E \vdash A \leftrightarrow A' \quad E, x:A \vdash b \leftrightarrow b' : B}{E \vdash \lambda(x:A)b \leftrightarrow \lambda(x:A')b' : A \rightarrow B} \qquad \frac{E \vdash b \leftrightarrow b' : A \rightarrow B \quad E \vdash a \leftrightarrow a' : A}{E \vdash b(a) \leftrightarrow b'(a') : B} \\
\\
\text{(Val Eq Abs2)} \\
\frac{E \vdash K \leftrightarrow K' \text{ kind} \quad E \vdash A \leftrightarrow A' :: K \quad E, X<:A::K \vdash b \leftrightarrow b' : B}{E \vdash \lambda(X<:A::K)b \leftrightarrow \lambda(X<:A'::K')b' : \forall(X<:A::K)B} \\
\\
\text{(Val Eq Appl2)} \\
\frac{E \vdash b \leftrightarrow b' :: \forall(X<:A''::K)B\{X\} \quad E \vdash A <: A'' :: K \quad E \vdash A \leftrightarrow A' :: K}{E \vdash b(A) \leftrightarrow b'(A') : B\{A\}} \\
\\
\text{(Val Eq Fold)} \\
\frac{E \vdash A \leftrightarrow \mu(X)B\{X\} \quad E \vdash A \leftrightarrow A' \quad E \vdash a \leftrightarrow a' : B\{A\}}{E \vdash \text{fold}(A,a) \leftrightarrow \text{fold}(A',a') : A} \\
\\
\text{(Val Eq Unfold)} \qquad \qquad \qquad \text{(Val Eq Rec)} \\
\frac{E \vdash a \leftrightarrow a' : \mu(X)B\{X\}}{E \vdash \text{unfold}(a) \leftrightarrow \text{unfold}(a') : \mu(X)B\{X\}} \qquad \frac{E \vdash A \leftrightarrow A' \text{ type} \quad E, x:A \vdash b \leftrightarrow b' : A}{E \vdash \mu(x:A)b \leftrightarrow \mu(x:A')b' : A} \\
\\
\text{(Val Eq Eta2)} \qquad \qquad \qquad \text{(Val Eq Eta)} \\
\frac{E \vdash b : \forall(X<:A::K)B \quad X \notin \text{dom}(E)}{E \vdash (\lambda(X<:A::K)b(X)) \leftrightarrow b : \forall(X<:A::K)B} \qquad \frac{E \vdash b : A \rightarrow B \quad x \notin \text{dom}(E)}{E \vdash (\lambda(x:A)b(x)) \leftrightarrow b : A \rightarrow B} \\
\\
\text{(Val Eq Beta2)} \qquad \qquad \qquad \text{(Val Eq Beta)} \\
\frac{E \vdash (\lambda(X<:A::K)b\{X\})(A) : B}{E \vdash (\lambda(X<:A::K)b)(A) \leftrightarrow b\{A\} : B} \qquad \frac{E \vdash (\lambda(x:A)b\{x\})(a) : B}{E \vdash (\lambda(x:A)b)(a) \leftrightarrow b\{a\} : B} \\
\\
\text{(Val Eq Fold-Unfold)} \\
\frac{E \vdash a : B\{\mu(X)B\{X\}\}}{E \vdash \text{unfold}(\text{fold}(\mu(X)B\{X\},a)) \leftrightarrow a : B\{\mu(X)B\{X\}\}} \\
\\
\text{(Val Eq Unfold-Fold)} \qquad \qquad \qquad \text{(Val Eq Fix)} \\
\frac{E \vdash a : \mu(X)B\{X\}}{E \vdash \text{fold}(\mu(X)B\{X\}, \text{unfold}(a)) \leftrightarrow a : \mu(X)B\{X\}} \qquad \frac{E \vdash \mu(x:A)b\{x\} : A}{E \vdash \mu(x:A)b \leftrightarrow b\{\mu(x:A)b\} : A}
\end{array}$$