Draft

Notes about $F^{\omega}_{<:}$

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$F^{\omega}_{<:}$

This is the generalization of $F_{<:}$ to ω-order, plus monotonicity.

K ::=	Kinds
TY	the kind of all types
$P_K(A)$	the kind of all subfamilies of a family
$\Pi(X::K)K'$	the kind of operators between kinds
$\Pi^+(X::K)K'$	the kind of monotonic operators
$\Pi^-(X::K)K'$	the kind of antimonotonic operators

A ::=	Families (Types and Operators)
X	type variables
Top	the supertype of all types
$A \to A'$	function spaces
$\forall(X::K)A'$	bounded quantifications
$\Lambda(X::K)A$	operators
A(A')	operator applications

a ::=	Values
x	value variables
top	canonical value of type Top
$\lambda(x:A)a$	functions
a(a')	applications
$\lambda(X::K)a$	bounded type functions
a(A)	type applications

Environments

In $F_{<:}$ $\varnothing$ E,x:A E,X<:A

In $F^{\omega}_{<:}$ $\varnothing$ E,x:A E,X::K

Judgments

In $F_{<:}$
$E \vdash A \text{ type} \equiv E \vdash A :: TY$
$E \vdash A <: B \equiv E \vdash A :: P_{TY}(B)$

In $F^{\omega}_{<:}$
$E \vdash A :: K$

Notation

$A<:B::K \equiv A::P_K(B)$

Note

$TY \approx P_{TY}(Top)$

Judgments

$\vdash E$
$E \vdash K$ kind
$E \vdash A::K$
$E \vdash a:A$
$E \vdash K<::L$
$E \vdash A<:B :: K$
$E \vdash A=B :: K$
$E \vdash a=b : A$

Rules

$$\frac{\vdash E,X{::}P_K(A),E'\ \mathrm{env}}{E,X{::}P_K(A),E' \vdash X{::}K}$$

$$\frac{\vdash E,X{::}K,E'\ \mathrm{env}}{E,X{::}K,E' \vdash X{::}K}$$

(Π I)

$$\frac{E,X{::}K \vdash B{::}L}{E \vdash \Lambda(X{::}K)B :: \Pi(X{::}K)L}$$

(Π E)

$$\frac{E \vdash B{::}\Pi(X{::}K)L \quad E \vdash A{::}K}{E \vdash B(A){::}L\{X\leftarrow A\}}$$

(Π $^+$ I)

$$\frac{E \vdash B{::}\Pi(X{::}K)L \qquad E,Y'{::}K,Y{<:}Y'{::}K \vdash B(Y){<:}B(Y'){::}L}{E \vdash B :: \Pi^+(X{::}K)L}$$

(Π $^+$ E)

$$\frac{E \vdash B{::}\Pi^+(X{::}K)L \quad E \vdash A{::}K}{E \vdash B(A) :: L\{X\leftarrow A\}}$$

(Π $^-$ I)

$$\frac{E \vdash B{::}\Pi(X{::}K)L \qquad E,Y'{::}K,Y{<:}Y'{::}K \vdash B(Y'){<:}B(Y){::}L}{E \vdash B :: \Pi^-(X{::}K)L}$$

(Π $^-$ E)

$$\frac{E \vdash B{::}\Pi^-(X{::}K)L \quad E \vdash A{::}K}{E \vdash B(A) :: L\{X\leftarrow A\}}$$

(Π <: I)

$$\frac{E,X{::}K \vdash B{<:}B'{::}L}{E \vdash \Lambda(X{::}K)B <: \Lambda(X{::}K)B' :: \Pi(X{::}K)L}$$

(Π <: E)

$$\frac{E \vdash B{<:}B'{::}\Pi(X{::}K)L \quad E \vdash A{::}K}{E \vdash B(A) <: B'(A) :: L\{X\leftarrow A\}}$$

(Π $^+$ <: I)

$$\frac{E \vdash B{<:}B'{::}\Pi(X{::}K)L \qquad E \vdash B :: \Pi^+(X{::}K)L \quad E \vdash B' :: \Pi^+(X{::}K)L}{E \vdash B <: B' :: \Pi^+(X{::}K)L}$$

(Π $^+$ <: E)

$$\frac{E \vdash B{::}\Pi^+(X{::}K)L \quad E \vdash A{<:}A'{::}K}{E \vdash B(A) <: B(A') :: L\{X\leftarrow A\}}$$

(Π $^-$ <: I)

$$\frac{E \vdash B{<:}B'{::}\Pi(X{::}K)L \qquad E \vdash B :: \Pi^-(X{::}K)L \quad E \vdash B' :: \Pi^-(X{::}K)L}{E \vdash B <: B' :: \Pi^-(X{::}K)L}$$

(Π $^-$ <: E)

$$\frac{E \vdash B{::}\Pi^-(X{::}K)L \quad E \vdash A{<:}A'{::}K}{E \vdash B(A') <: B(A) :: L\{X\leftarrow A\}}$$

---- beta, eta

3. Operator fragment

(A simplified version of the previous rules where $X \notin FV(L)$, etc.)

$$\frac{\vdash E,X<:A::K,E'\ \mathrm{env}}{E,X<:A::K,E' \vdash X::K}$$

$$\frac{\vdash E,X<:A::K,E'\ \mathrm{env}}{E,X<:A::K,E' \vdash X<:A::K}$$

($\Rightarrow$ I)

$$\frac{E,X::K \vdash B::L}{E \vdash \Lambda(X::K)B :: K\Rightarrow L}$$

($\Rightarrow$ E)

$$\frac{E \vdash B::K\Rightarrow L \quad E \vdash A::K}{E \vdash B(A)::L}$$

($\Rightarrow^{+}$ I)

$$\frac{\begin{array}{c} E \vdash B::K\Rightarrow L \\ E,Y'::K,Y<:Y'::K \vdash B(Y)<:B(Y')::L \end{array}}{E \vdash B :: K\Rightarrow^{+}L}$$

($\Rightarrow^{+}$ E)

$$\frac{E \vdash B::K\Rightarrow^{+}L \quad E \vdash A::K}{E \vdash B(A) :: L}$$

($\Rightarrow^{-}$ I)

$$\frac{\begin{array}{c} E \vdash B::K\Rightarrow L \\ E,Y'::K,Y<:Y'::K \vdash B(Y')<:B(Y)::L \end{array}}{E \vdash B :: K\Rightarrow^{-}L}$$

($\Rightarrow^{-}$ E)

$$\frac{E \vdash B::K\Rightarrow^{-}L \quad E \vdash A::K}{E \vdash B(A) :: L}$$

($\Rightarrow$ <: I)

$$\frac{E,X::K \vdash B<:B'::L}{E \vdash \Lambda(X::K)B <: \Lambda(X::K)B' :: K\Rightarrow L}$$

($\Rightarrow$ <: E)

$$\frac{E \vdash B<:B'::K\Rightarrow L \quad E \vdash A::K}{E \vdash B(A) <: B'(A) :: L}$$

($\Rightarrow^{+}$ <: I)

$$\frac{\begin{array}{c} E \vdash B<:B'::K\Rightarrow L \\ E \vdash B :: K\Rightarrow^{+}L \quad E \vdash B' :: K\Rightarrow^{+}L \end{array}}{E \vdash B <: B' :: K\Rightarrow^{+}L}$$

($\Rightarrow^{+}$ <: E)

$$\frac{E \vdash B::K\Rightarrow^{+}L \quad E \vdash A<:A'::K}{E \vdash B(A) <: B(A') :: L}$$

($\Rightarrow^{-}$ <: I)

$$\frac{\begin{array}{c} E \vdash B<:B'::K\Rightarrow L \\ E \vdash B :: K\Rightarrow^{-}L \quad E \vdash B' :: K\Rightarrow^{-}L \end{array}}{E \vdash B <: B' :: K\Rightarrow^{-}L}$$

($\Rightarrow^{-}$ <: E)

$$\frac{E \vdash B::K\Rightarrow^{-}L \quad E \vdash A<:A'::K}{E \vdash B(A') <: B(A) :: L}$$

(beta)

$$\frac{E, X::K \vdash B = B' :: L \quad E \vdash A = A' :: K}{E \vdash (\Lambda(X::K)B)(A) = B'\{X\leftarrow A'\} :: L}$$

(eta)

$$\frac{E \vdash B = B'::K\Rightarrow L \quad X\notin dom(E)}{E \vdash \Lambda(X::K)B(X) = B' :: K\Rightarrow L}$$

4. Lemmas

Prop

$$\frac{E \vdash F::K \Rightarrow^{+}(L \Rightarrow M) \qquad E \vdash F::K \Rightarrow (L \Rightarrow^{+} M)}{E \vdash F::K \Rightarrow^{+}(L \Rightarrow^{+} M)}$$

Proof

$$\frac{\begin{array}{l} E \vdash F::K \Rightarrow (L \Rightarrow^{+} M) \\ E \vdash F::K \Rightarrow (L \Rightarrow^{+} M) \end{array}}{E,Y'::K,Y<:Y'::K \vdash F(Y)::L \Rightarrow^{+} M \qquad E,Y'::K,Y<:Y'::K \vdash F(Y')::L \Rightarrow^{+} M} \ \text{(weaken) } (\Rightarrow E)$$

1)

$$\frac{\dfrac{\dfrac{E \vdash F::K \Rightarrow^{+}(L \Rightarrow M)}{E,Y'::K,Y<:Y'::K \vdash F(Y)<:F(Y')::L \Rightarrow M} \ \text{(weaken) } (\Rightarrow^{+} <: E) \qquad (1)}{E,Y'::K,Y<:Y'::K \vdash F(Y)<:F(Y')::L \Rightarrow^{+} M} \ (\Rightarrow^{+} <: I) \qquad E \vdash F::K \Rightarrow (L \Rightarrow^{+} M)}{E \vdash E \vdash F::K \Rightarrow^{+}(L \Rightarrow^{+} M)} \ (\Rightarrow^{+} I)$$

Prop

$(\Rightarrow^{+} I\ 2)$

$$\frac{\begin{array}{c} E \vdash B::K \Rightarrow L \Rightarrow M \\ E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y')(Z)::M \\ E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y)(Z')::M \end{array}}{E \vdash B :: K \Rightarrow^{+} L \Rightarrow^{+} M}$$

Proof

$$\dfrac{\dfrac{\dfrac{E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y')(Z)::M}{E,Y'::K,Y<:Y'::K \vdash \lambda(Z::L)B(Y)(Z)<:\lambda(Z::L)B(Y')(Z)::L \Rightarrow M}}{E,Y'::K,Y<:Y'::K \vdash B(Y)<:B(Y')::L \Rightarrow M}}{E \vdash B::K \Rightarrow^{+} L \Rightarrow M}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{E,Y'::K,Y<:Y'::K,Z'::L,Z<:Z'::L \vdash B(Y)(Z)<:B(Y)(Z')::M}{E,Y'::K,Y<:Y'::K \vdash B(Y)::L \Rightarrow^{+} M}}{E,Y'::K \vdash \lambda(Y<:Y'::K)B(Y)::\Pi(Y<:Y'::K)L \Rightarrow^{+} M}}{E,Y'::K \vdash B(Y')::L \Rightarrow^{+} M}}{E \vdash \lambda(Y'::K)B(Y')::K \Rightarrow L \Rightarrow^{+} M}}{E \vdash B::K \Rightarrow L \Rightarrow^{+} M}$$

Hence $E \vdash B :: K \Rightarrow^{+} L \Rightarrow^{+} M$ by the previous proposition.

5. The kind NAT in $F^{\omega}_{<:}$

By analogy with Church numerals:

$$\text{NAT' kind} \triangleq (\text{TY}\Rightarrow\text{TY}) \Rightarrow (\text{TY}\Rightarrow\text{TY})$$

$$\text{Zero} :: \text{NAT'} \triangleq \Lambda(\text{S}::\text{TY}\Rightarrow\text{TY})\, \Lambda(\text{Z}::\text{TY})\, \text{Z}$$

$$\text{Succ} :: \text{NAT'}\Rightarrow \text{NAT'} \triangleq$$

$$\Lambda(\text{N}::\text{NAT})\, \Lambda(\text{S}::\text{TY}\Rightarrow\text{TY})\, \Lambda(\text{Z}::\text{TY})\, \text{S}(\text{N}(\text{S})(\text{Z}))$$

Suppose now we want Succ to be monotonic. We are forced into the following definition, where all $\Rightarrow^{+}$ are necessary:

$$\text{NAT kind} \triangleq (\text{TY}\Rightarrow^{+}\text{TY}) \Rightarrow^{+} (\text{TY}\Rightarrow^{+}\text{TY})$$

$$\text{Zero} :: \text{NAT'} \triangleq \Lambda(\text{S}::\text{TY}\Rightarrow^{+}\text{TY})\, \Lambda(\text{Z}::\text{TY})\, \text{Z}$$

$$\text{Succ} :: \text{NAT}\Rightarrow \text{NAT'} \triangleq$$

$$\Lambda(\text{N}::\text{NAT})\, \Lambda(\text{S}::\text{TY}\Rightarrow^{+}\text{TY})\, \Lambda(\text{Z}::\text{TY})\, \text{S}(\text{N}(\text{S})(\text{Z}))$$

It is easy to show that:

$$\text{Zero} :: \text{NAT}$$

$$\text{Succ} :: \text{NAT}\Rightarrow\text{NAT}$$

The we can prove:

$$\text{A}<:\text{B}::\text{NAT} \supset \text{A}(\text{S})(\text{Z})<:\text{B}(\text{S})(\text{Z})::\text{TY} \quad \text{for } \text{S}::\text{TY}\Rightarrow^{+}\text{TY},\ \text{Z}::\text{TY}$$

$$\supset \Lambda(\text{Z}::\text{TY})\text{A}(\text{S})(\text{Z}) <: \Lambda(\text{Z}::\text{TY})\text{B}(\text{S})(\text{Z}) :: \text{TY}\Rightarrow^{+}\text{TY}$$

$$\supset \Lambda(\text{S}::\text{TY}\Rightarrow^{+}\text{TY})\Lambda(\text{Z}::\text{TY})\text{A}(\text{S})(\text{Z}) <: \Lambda(\text{S}::\text{TY}\Rightarrow^{+}\text{TY})\Lambda(\text{Z}::\text{TY})\text{B}(\text{S})(\text{Z}) :: \text{NAT}$$

$$\supset \text{Succ}(\text{A}) <: \text{Succ}(\text{B}) :: \text{NAT}$$

That is, $\text{Succ} :: \text{NAT}\Rightarrow^{+}\text{NAT}$

Morale: in order to obtain monotonic constructors, the iterator of kind NAT must be monotonic in all positions.

6. Pairs of types in $F^{\omega}_{<:}$

PAIR' kind $\triangleq$ $(TY \Rightarrow^{+} TY \Rightarrow^{+} TY) \Rightarrow TY$

PAIR kind $\triangleq$ $(TY \Rightarrow^{+} TY \Rightarrow^{+} TY) \Rightarrow^{+} TY$

Pair :: $TY \Rightarrow TY \Rightarrow$ PAIR'
$\triangleq$ $\Lambda(X::TY)\ \Lambda(Y::TY)\ \Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(X)(Y)$

Prop: Pair :: $TY \Rightarrow TY \Rightarrow$ PAIR
Proof
Assume A,B::TY
Assume $Y <: Y' :: TY \Rightarrow^{+} TY \Rightarrow^{+} TY$
Then $Y(A) <: Y'(A) :: TY \Rightarrow^{+} TY$ $(\Rightarrow^{+} <: E)$
Then $Y(A)(B) <: Y'(A)(B) :: TY$ $(\Rightarrow^{+} <: E)$
Then $(\Lambda(X::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ X(A)(B))(Y)$
$<: (\Lambda(X::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ X(A)(B))(Y') :: TY$ (β)
I.e.: $Pair(A)(B)(Y) <: Pair(A)(B)(Y') :: TY$
Hence $Pair(A)(B) :: (TY \Rightarrow^{+} TY \Rightarrow^{+} TY) \Rightarrow^{+} TY = PAIR$ $(\Rightarrow^{+} I)$
Finally $\Lambda(A::TY)\ \Lambda(B::TY)\ Pair(A)(B) :: TY \Rightarrow TY \Rightarrow PAIR$ $(\Rightarrow I)$
and $Pair :: TY \Rightarrow TY \Rightarrow PAIR$ (η)

Prop: Pair :: $TY \Rightarrow TY \Rightarrow^{+}$ PAIR
Proof
Assume A::TY, B<:B'
Assume $W::TY \Rightarrow^{+} (TY \Rightarrow^{+} TY)$
Hence $W(A)(B) <: W(A)(B') :: TY$ $(\Rightarrow^{+} <: E)$
Hence $(\Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(A)(B))(W)$
$<: (\Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(A)(B'))(W) :: TY$ (β)
I.e.: $Pair(A)(B)(W) <: Pair(A)(B')(W) :: TY$
Then $\Lambda(W::TY \Rightarrow^{+} TY \Rightarrow^{+} TY) Pair(A)(B)(W)$
$<: \Lambda(W::TY \Rightarrow^{+} TY \Rightarrow^{+} TY) Pair(A)(B')(W) :: PAIR'$ $(\Rightarrow <: I)$
I.e.: $Pair(A)(B) <: Pair(A)(B') :: PAIR'$ (η)
But $Pair(A)(B), Pair(A)(B') :: PAIR$ (previous prop)
Hence $Pair(A)(B) <: Pair(A)(B') :: PAIR$ $(\Rightarrow^{+} <: I)$
Finally $Pair(A) :: TY \Rightarrow^{+} PAIR$ $(\Rightarrow^{+} I)$
and $Pair :: TY \Rightarrow TY \Rightarrow^{+} PAIR$ $(\Rightarrow I)\ (\eta)$

Prop: Pair :: $TY \Rightarrow^{+} TY \Rightarrow^{+} PAIR$
Proof
  Assume $A <: A'$, $B :: TY$
  Assume $W :: TY \Rightarrow^{+} TY \Rightarrow^{+} TY$
  Hence $W(A)(B) <: W(A')(B) :: TY$ $(\Rightarrow^{+} <: E)$
  Hence $(\Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(A)(B))(W)$
      $<: (\Lambda(Z::TY \Rightarrow^{+} TY \Rightarrow^{+} TY)\ Z(A')(B))(W) :: TY$ (β)
  I.e.: $Pair(A)(B)(W) <: Pair(A')(B)(W) :: TY$
  Then $\Lambda(W::TY \Rightarrow^{+} TY \Rightarrow^{+} TY) Pair(A)(B)(W)$
    $<: \Lambda(W::TY \Rightarrow^{+} TY \Rightarrow^{+} TY) Pair(A')(B)(W) :: PAIR'$ $(\Rightarrow <: I)$
  I.e.: $Pair(A)(B) <: Pair(A')(B) :: PAIR'$ (η)
  But $Pair(A)(B), Pair(A')(B) :: PAIR$ (previous prop)
  Hence $Pair(A)(B) <: Pair(A')(B) :: PAIR$ $(\Rightarrow^{+} <: I)$
  Then $\Lambda(B::TY) Pair(A)(B)$
    $<: \Lambda(B::TY) Pair(A')(B) :: TY \Rightarrow PAIR$ $(\Rightarrow <: I)$
  I.e.: $Pair(A) <: Pair(A') :: TY \Rightarrow PAIR$ (η)
  But $Pair(A), Pair(A') :: TY \Rightarrow^{+} PAIR$ (previous prop)
  Hence $Pair(A) <: Pair(A') :: TY \Rightarrow^{+} PAIR$ $(\Rightarrow^{+} <: I)$
  Finally $Pair <: Pair :: TY \Rightarrow^{+} TY \Rightarrow^{+} PAIR$ $(\Rightarrow^{+} I)$

+ Cor: If $A <: A'$, $B <: B'$ then $Pair(A)(B) <: Pair(A')(B') :: PAIR$
Proof From $Pair :: TY \Rightarrow^{+} TY \Rightarrow^{+} PAIR$ by $(\Rightarrow^{+} <: E)$

$Fst :: PAIR \Rightarrow TY$
  $\triangleq\ \Lambda(P::PAIR)\ P(\Lambda(X::TY)\ \Lambda(Y::TY)\ X)$

Prop: $Fst :: PAIR \Rightarrow^{+} TY$
  (If $P <: P' :: PAIR$ then $Fst(P) <: Fst(P')$)
Proof
  Assume $P <: P' :: PAIR$
  Note that $\Lambda(X::TY)\ \Lambda(Y::TY)\ X :: TY \Rightarrow^{+} TY \Rightarrow^{+} TY$
  Then $P(\Lambda(X::TY)\ \Lambda(Y::TY)\ X)$
    $<: P'(\Lambda(X::TY)\ \Lambda(Y::TY)\ X) :: TY$ $(\Rightarrow^{+} <: E)$
  Then $(\Lambda(P::PAIR)\ P(\Lambda(X::TY)\ \Lambda(Y::TY)\ X))(P)$
    $<: (\Lambda(P::PAIR)\ P(\Lambda(X::TY)\ \Lambda(Y::TY)\ X))(P') :: TY$ (β)
  I.e.: $Fst(P) <: Fst(P') :: TY$
  Hence $Fst :: PAIR \Rightarrow^{+} TY$ $(\Rightarrow^{+} I)$

$Snd :: PAIR \Rightarrow TY$
  $\triangleq\ \Lambda(P::PAIR)\ P(\Lambda(X::TY)\ \Lambda(Y::TY)\ Y)$

Prop: Snd :: $PAIR \Rightarrow^{+} TY$
(If $P <: P' :: PAIR$ then $Snd(P) <: Snd(P')$)

Prod :: $PAIR \Rightarrow TY$
$\triangleq$ $\Lambda(P::PAIR)\ Fst(P) \times Snd(P)$

Prop: Prod :: $PAIR \Rightarrow^{+} TY$
(If $P <: P' :: PAIR$ then $Prod(P) <: Prod(P') :: TY$)

Proof
Assume $Q <: Q' :: PAIR$
By hyp and previous facts: $Fst(Q) <: Fst(Q')$; $Snd(Q) <: Snd(Q')$
By monotonicity of $\times$: $Fst(Q) \times Snd(Q) <: Fst(Q') \times Snd(Q')$
Expansion: $(\Lambda(P::PAIR)\ Fst(P) \times Snd(P))(Q)$
$<: (\Lambda(P::PAIR)\ Fst(P) \times Snd(P))(Q')$
I.e.: $Prod(Q) <: Prod(Q')$
Hence Prod :: $PAIR \Rightarrow^{+} TY$ $(\Rightarrow^{+} I)$