Modular Markovian Logic

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Abstract. We introduce Modular Markovian Logic (MML) for compositional continuous-time and continuous-space Markov processes. MML combines operators specific to stochastic logics with operators that reflect the modular structure of the semantics, similar to those used by spatial and separation logics. We present a complete Hilbert-style axiomatization for MML, prove the small model property and analyze the relation between the stochastic bisimulation and the logical equivalence relation induced by MML on models.

1 Introduction

Complex networks (e.g., embedded systems, communication networks, the Internet etc.) and complex systems (e.g., biological, ecological, social, financial, etc.) are often modelled as stochastic processes, to encapsulate a lack of knowledge or inherent randomness. Such systems are frequently modular in nature, consisting of parts which are systems in their own right. Their global behaviour depends on the behaviour of their parts and on the links which connect them. Understanding such systems requires integration of local stochastic information in a formal way, in order to address questions such as: *"to what extent is it possible to derive global properties of the system from the local properties of its modules?"*.

This is a problem of fundamental importance in complex systems that has been usually addressed semantically: probabilistic and stochastic process algebras [3], for instance, aim at describing compositionally the behaviour of a system from the behaviours of its susbsystems taking into account various types of synchronization or communication. This approach is quite restrictive, as process algebras are not logics: one cannot express basic logical operations such as conjunction, disjunction, implication or negation of properties. Usually, to do this, people use logics such as temporal logics [15], modal μ-calculus [22] or Hennessy-Milner logic [20] to express properties of transition systems. But these are global properties only and no logic framework developed so far allows reasoning on stochastic systems and subsystems at the same time.

In this paper we develop a logical framework called Modular Markovian Logic (MML) that tackles this problem by organizing qualitative and quantitative properties of stochastic systems in hierarchical, modular structures, thereby proving global properties from the local properties of modules. Formally, if *"process P has the property ϕ"* is denoted by $P \Vdash \phi$ and "$\otimes$" is the composition operator, we aim to establish a framework

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containing modular proof rules of the form $\dfrac{P_1 \Vdash \phi_1, ..., P_k \Vdash \phi_k}{P_1 \otimes ... \otimes P_k \Vdash \rho} C(\rho, \phi_1, .., \phi_k)$, where C is a logical constraint.

To gain this level of expressivity, MML combines stochastic operators similar to the ones of Aumann's system [1, 16] with modular operators similar to the ones used in spatial logics [6, 7] and in separation logics [31]. For an observable action a and a positive rational r, the operator "L_r^a" of MML expresses the fact that a process can perform an a-transition with the rate[1] at least r. In addition, the *composition operator* "|" joins logical terms and directly expresses properties of the combined subsystems, and dually, the *quotient operator* " $-$ " quantifies over possible environments satisfying given specifications.

On the semantic level, we introduce the *modular Markov processes* (MMPs) which are (continuous-) labelled Markov processes [14, 29] enriched with an algebraic structure. This algebra defines the composition of Markovian systems and establishes the relation between a system and its subsystems. The composition of behaviours satisfies a general synchronization pattern which subsumes most of the classical notions of parallel composition found in process algebras.

We define the modular Markovian logic for a semantics based on MMPs. We investigate the relation between stochastic bisimulations of MMPs and logical equivalence induced by MML over the class of MMPs. We present a complete Hilbert style axiomatization of MML for the Markovian semantics and prove the small model property.

Research context. Labelled Markov process (LMPs) are introduced in [13, 4, 14, 29] and they generalize most of the models of Markovian systems. A similar concept, Harsanyi type space (HTS), has been studied in the context of belief systems [17, 28]. MMPs are built on top of these, by exploiting their equivalence proved in [11]. In addition, MMPs have inbuilt an algebraic structure that exyends, for continuous space and time, the concepts of the Markov chain algebra [5].

Probabilistic logics have been studied both for LMPs (probabilistic versions of temporal and Hennessy-Milner logics [14, 12, 29]) and for HTSs (Aumann's system [1, 16]). The first class focuses on model checking and logical characterization of stochastic bisimulation, while for Aumann's system also axiomatization issues have been addressed [19, 32]. In [10] we have proposed a completely axiomatized stochastic logic that combines features of the two classes of logics. In this paper we extend the stochastic logic with modular operators that allow us, in addition, to investigate the algebraic structures of the models.

Modular logics, such as spatial logics [6, 7] and separation logic [31] have been developed for concurrent nondeterministic systems, but to the best of our knowledge, no stochastic or probabilistic version of these have been studied.

The paper is organized as follows. Section 2 introduces basic concepts used in the paper. Section 3 defines MMPs and their bisimulation. Section 4 presents MML and results concerning the relationship between logical equivalence and bisimulation. Section 5 contains the axiomatic system of MML, the soundness and completeness metatheo-

[1] The rate of a transition is the parameter of an exponentially distributed random variable that characterizes, for Markovian processes, the duration of the transition.

rems and the small model property. In addition to the conclusive remarks, the paper has an Appendix containing some proofs which have not been included in the paper.

2 Preliminary definitions

In this section we establish the terminology used in the paper.

Given a set M, $\Sigma \subseteq 2^M$ that contains M and is closed under complement and countable union is a *σ-algebra* over M; (M, Σ) is a *measurable space* and the elements of Σ are *measurable sets*. $\Omega \subseteq 2^M$ is a *base for Σ* if Σ is the closure of Ω under complement and countable union; we write $\overline{\Omega} = \Sigma$.

A relation $\mathfrak{R} \subseteq M \times M$ is *non-wellfounded* if there exists $\{m_i \in M \mid i \in \mathbb{N}\}$ such that for each $i \in \mathbb{N}$, $(m_i, m_{i+1}) \in \mathfrak{R}$; otherwise it is *wellfounded*. A subset $N \subseteq M$ is $\mathfrak{R}$-closed iff $\{m \in M \mid \exists n \in N, (m, n) \in \mathfrak{R}\} \subseteq N$. If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

A *measure* on (M, Σ) is a function $\mu : \Sigma \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$ and for $\{N_i | i \in I \subseteq \mathbb{N}\} \subseteq \Sigma$ with pairwise disjoint elements, $\mu(\bigcup_{i \in I} N_i) = \sum_{i \in I} \mu(N_i)$.

Let $\Delta(M, \Sigma)$ be the class of measures on (M, Σ). We organize it as a measurable space by considering the σ-algebra generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets $\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$.

Given two measurable spaces (M, Σ) and (N, Θ), a mapping $f : M \to N$ is *measurable* if for any $T \in \Theta$, $f^{-1}(T) \in \Sigma$. We use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Θ).

Central for this paper is the notion of an *analytic set*. We only recall the main definition and mention the properties of analytic sets used in our proves. For detailed discussion on this topic related to Markov processes, the reader is referred to [29] (Section 7.5) or to [11] (Section 4.4).

A metric space (M, d) is *complete* if every Cauchy sequence converges in M.

A *Polish space* is the topological space underlying a complete metric space with a countable dense subset. Note that any discrete space is Polish.

An *analytic set* is the image of a Polish space under a continuous function between Polish spaces. Note that any Polish space is an analytic set.

There are some basic facts about analytic sets that we use in this paper. Firstly, an analytic set, as measurable space, has a denumerable base with disjoint elements. Secondly, If $\mathcal{M}_1, \mathcal{M}_2$ are analytic sets with Σ_1, Σ_2 the Borel algebras generated by their topologies, then the product space $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with the Borel algebra Σ generated by the product topology is an analytic set.

3 Modular Markov processes

For the beginning we introduce continuous Markov processes (CMPs) for a finite set $\mathcal{A}$ of *actions*. CMPs are coalgebraic structures that encode stochastic behaviors. If m is the current state of the system, N a measurable set of states and $a \in \mathcal{A}$, $\theta(a)(m)$ is a measures on the state space and $\theta(a)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable that characterizes the duration of an a-transition from m

to arbitrary $n \in N$. Indeterminacy is resolved by races between events executing at different rates.

Definition 1 (Continuous Markov processes). *Given an analytic set* (M, Σ)*, where* Σ *is the Borel algebra generated by the topology, an* $\mathcal{A}$-continuous Markov kernel *is a tuple* $\mathcal{K} = (M, \Sigma, \theta)$*, where* $\theta : \mathcal{A} \to [\![M \to \Delta(M, \Sigma)]\!]$.
If $m \in M$*,* $(\mathcal{K}, m)$ *is an* $\mathcal{A}$-continuous Markov process[2].

Let $\mathfrak{K}$ be the class of $\mathcal{A}$-CMKs; $\mathcal{K}, \mathcal{K}_i, \mathcal{K}'$ are used to range over $\mathcal{A}$-CMKs. Stochastic bisimulation follows the line of Larsen-Skou bisimulation [25, 12, 29].

Definition 2 (Stochastic Bisimulation). *Given* $\mathcal{K} = (M, \Sigma, \theta) \in \mathfrak{K}$*, a* rate-bisimulation relation *on* $\mathcal{K}$ *is a relation* $\mathfrak{R} \subseteq M \times M$ *such that* $(m, n) \in \mathfrak{R}$ *iff for any* $C \in \Sigma(\mathfrak{R})$ *and any* $a \in \mathcal{A}$*,* $\theta(a)(m)(C) = \theta(a)(n)(C)$.
Two processes $(\mathcal{K}, m)$ *and* $(\mathcal{K}, n)$ *are* stochastic bisimilar*, written* $m \sim_{\mathcal{K}} n$*, if they are related by a rate-bisimulation relation.*

Two processes $(\mathcal{K}, m)$ and $(\mathcal{K}', m')$ are *stochastic bisimilar*, written $(\mathcal{K}, m) \sim (\mathcal{K}', m')$, iff $m \sim_{\mathcal{K} \uplus \mathcal{K}'} m'$, where $\mathcal{K} \uplus \mathcal{K}'$ is the disjoint union of $\mathcal{K}$ and $\mathcal{K}'$.

3.1 Synchronization

To define the modular Markov processes we need a general notion of synchronization of CMPs. For this, we assume extra structure on the set $\mathcal{A}$ of actions.

Firstly, we consider a *synchronisation function* $*$ that is a partial function $* : \mathcal{A} \times \mathcal{A} \hookrightarrow \mathcal{A}$ which associates to some $a, b \in \mathcal{A}$ an action $a * b \in \mathcal{A}$ interpreted as the synchronisation of a and b. In this way we can mimic various synchronisation paradigms: for CCS-style [27] we ask that $a * \overline{a} = \tau$, where $\tau \in \mathcal{A}$ is a special action; for CSP-style [21] we ask that $a * a = a$; for interleaving and ACP-style [2] we assume that there exists a reflexive transition $\delta \in \mathcal{A}$ such that $a * \delta = a$ for any $a \in \mathcal{A}$. Similarly, most classical notions of parallel composition in process algebras may be expressed by a suitable synchronization function. In fact, the only formal requirement is that $*$, as an operation, is *commutative* ($a * b = b * a$).

Secondly, we assume a function $\bullet : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that computes, given the rates r and s of the actions a and b respectively, the rate $r \bullet s$ of the synchronisation $a * b$. Examples of such function are the *mass action law* used with stochastic Pi-calculus [30, 9] and other models of bio-chemical interactions and the *minimal rate law* used by PEPA [18] for applications in performance evaluation. The formal requirements are:

$\bullet : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *continuous* function that, as an operation, is *commutative* ($r \bullet s = s \bullet r$), *associative* ($(r \bullet s) \bullet t = r \bullet (s \bullet t)$) and *bilinear* ($(r_1 + r_2) \bullet s = (r_1 \bullet s) + (r_2 \bullet s)$ and $s \bullet (r_1 + r_2) = (s \bullet r_1) + (s \bullet r_2)$).

These two functions define the synchronization of two CMPs as follows.

[2] $\theta(\alpha)$ is a measurable mapping between (M, Σ) and $\Delta(M, \Sigma)$. This is equivalent with the conditions on the two-variable *rate function* used in [14] to define continuous Markov processes (see, e.g. Proposition 2.9, of [11]).

Definition 3. *For $i = 1, 2$, let $\mathcal{K}_i = (M_i, \Sigma_i, \theta_i) \in \mathfrak{K}$ and $\Delta_i \subseteq \Sigma_i$ denumerable bases with disjoint elements. $\mathcal{K} = (M, \Sigma, \theta)$ is the* product *of $\mathcal{K}_1$ and $\mathcal{K}_2$, written $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, if $M = M_1 \times M_2$, $\Sigma = \overline{\Sigma_1 \times \Sigma_2}$ and $\theta : \mathcal{A} \to [M \to [\Sigma \to \mathbb{R}_+]]$ is defined, for $m_i \in M_i$, $a \in \mathcal{A}$ and $S = \bigcup_{k \in K \subseteq \mathbb{N}} U_k^1 \times U_k^2 \in \Sigma$ for $U_k^i \in \Delta_i$, by*

$$\theta(a)((m_1, m_2))(S) = \sum_{(b,c) \in \mathcal{A}^2}^{b*c=a} \sum_{k \in K} \theta_1(b)(m_1)(U_k^1) \bullet \theta_2(c)(m_2)(U_k^2).$$

The properties of $\bullet$ guarantee that the previous sum is convergent and independent of the choice of the bases. Because $\bullet$ is bilinear, $r \bullet 0 = 0$. $\mathcal{K}$ represents the result of the synchronization of $\mathcal{K}_1$ and $\mathcal{K}_2$: θ calculates the rate of a by summing all the possible synchronizations $b * c = a$ between $\mathcal{K}_1$ and $\mathcal{K}_2$.

Lemma 1. *If $\mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{K}$, then $\mathcal{K}_1 \times \mathcal{K}_2 \in \mathfrak{K}$.*

If $(\mathcal{K}_1, m_1)$ and $(\mathcal{K}_2, m_2)$ are CMPs, then $(\mathcal{K}_1 \times \mathcal{K}_2, (m_1, m_2))$ is a CMP called the *synchronization of* $(\mathcal{K}_1, m_1)$ and $(\mathcal{K}_2, m_2)$.

3.2 Parallel composition

For introducing a concept of parallel composition that is general enough to include most of the similar concepts, we assume that the support set of the Markov kernel has an algebraic structure called *modular structure*.

Definition 4 (Modular structure). *A tuple $(M, \equiv, \otimes)$ is a* modular structure *on a set M if $\equiv \subseteq M \times M$ is an equivalence relation and $\otimes : M \times M \hookrightarrow M$ is a partial operation which, with respect to $\equiv$, is*
– a **congruence***, i.e., if $m_0 \equiv m_1$, then $m_0 \otimes m_2$ is defined iff $m_1 \otimes m_2$ is defined and $m_0 \otimes m_2 \equiv m_1 \otimes m_2$,*
– **associative***, i.e., $(m_0 \otimes m_1) \otimes m_2 \equiv m_0 \otimes (m_1 \otimes m_2)$,*
– **commutative***, i.e., $m_0 \otimes m_1 \equiv m_1 \otimes m_0$,*
– **modular***, i.e., if $m_0 \otimes m_1 \equiv n_0 \otimes n_1$, then for arbitrary $i, j \in \{0, 1\}$,*
either $m_i \equiv n_j$ and $m_{1-i} \equiv n_{1-j}$,
or there exists $m \in M$ such that $m_i \equiv n_j \otimes m$ and $n_{1-j} \equiv m_{1-i} \otimes m$;
– **wellfounded***, i.e., the relation $\{(m, n) \mid \exists n' \in M, m \equiv n \otimes n'\}$ is wellfounded.*

Process algebras are examples of modular structures where $\equiv$ is the structural congruence or some bisimulation relation, while $\otimes$ is, for instance, the parallel composition. In these cases wellfoundness expresses the fact that any process (modulo (Nil): $P \equiv P \otimes 0$) can be decomposed into a finite number of processes that cannot be, further, decomposed; and modularity guarantees the uniqueness of this decomposition up to structural congruence. In process algebras these hold, modulo (Nil), due to the inductive definition of the set of processes.

For modular structures, we lift the signature to sets by defining, for arbitrary $N, N' \subseteq M$, $N \otimes N' = \{m \in M \mid m \equiv n \otimes n'$ for some $n \in N, n' \in N'\}$ and $N \otimes^- N' = \{m \in M \mid \forall n' \in N'$ and $\forall n \equiv m \otimes n', n \in N\}$. Moreover, if $\Sigma \subseteq 2^M$, let $\Sigma \otimes \Sigma = \{N \otimes N' \mid N, N' \in \Sigma\}$ and $\Sigma \otimes^- \Sigma = \{N \otimes^- N' \mid N, N' \in \Sigma\}$.

Definition 5 (Modular Markov process). *An* $\mathcal{A}$-modular Markov kernel *is a tuple* $\mathcal{M} = (\mathcal{K}, \equiv, \otimes)$*, where* $\mathcal{K} = (M, \Sigma, \theta) \in \mathfrak{K}$ *and* $(M, \equiv, \otimes)$ *is a modular structure such that its algebraic structure*

• *preserves the Borel-algebras, in the sense that*

1. $\Sigma \otimes \Sigma \subseteq \Sigma$, 2. $\Sigma \otimes^{-} \Sigma \subseteq \Sigma$;

• *preserves the behaviours of modules and their synchronization, i.e.,*

3. $\equiv \subseteq \sim$, 4. $(\mathcal{K}, m_0 \otimes m_1) \sim (\mathcal{K} \times \mathcal{K}, (m_0, m_1))$.

If $m \in M$*,* $(\mathcal{M}, m)$ *is a* modular Markov process.

Condition 4 requires that $(\mathcal{K}, m_0 \otimes m_1)$ is bisimilar with the synchronization of $(\mathcal{K}, m_0)$ and $(\mathcal{K}, m_1)$.

M is called the *support of* $\mathcal{M}$, denoted $sup(\mathcal{M})$. Let $\mathfrak{M}$ be the class of $\mathcal{A}$-modular Markov kernels (MMKs); we use $\mathcal{M}, \mathcal{N}, \mathcal{M}_i, \mathcal{M}'$ to range over $\mathfrak{M}$.

Next we prove that for MMKs stochastic bisimulation is a congruence.

Theorem 1 (Congruence). *Given* $(\mathcal{K}, \equiv, \otimes) \in \mathfrak{M}$*, if* $m \sim_{\mathcal{K}} m'$ *and both* $m \otimes n$ *and* $m' \otimes n$ *are defined, then* $m \otimes n \sim_{\mathcal{K}} m' \otimes n$.

4 Modular Markovian Logic

In this section we introduce Modular Markovian Logic (MML).

The formulas of MML are the elements of the set $\mathcal{L}$ introduced by the following grammar, for arbitrary $a \in \mathcal{A}$ and $r \in \mathbb{Q}_+$.

$$\phi := \top \;\vdots\; \neg\phi \;\vdots\; \phi \wedge \phi \;\vdots\; L_r^a \phi \;\vdots\; \phi | \phi \;\vdots\; \phi - \phi.$$

The semantics is given by the *satisfiability relation* " $\Vdash$ " defined for $\mathcal{M} \in \mathfrak{M}$ and $m \in sup(\mathcal{M})$, inductively as follows.

$\mathcal{M}, m \Vdash \top$ always;

$\mathcal{M}, m \Vdash \neg\phi$ iff it is not the case that $\mathcal{M}, m \Vdash \phi$;

$\mathcal{M}, m \Vdash \phi \wedge \psi$ iff $\mathcal{M}, m \Vdash \phi$ and $\mathcal{M}, m \Vdash \psi$;

$\mathcal{M}, m \Vdash L_r^a \phi$ iff $\theta(a)(m)([\![\phi]\!]_{\mathcal{M}}) \geq r$, where $[\![\phi]\!]_{\mathcal{M}} = \{m \in M | \mathcal{M}, m \Vdash \phi\}$;

$\mathcal{M}, m \Vdash \phi_1 | \phi_2$ iff $m \equiv m_1 \otimes m_2$ and $\mathcal{M}, m_i \Vdash \phi_i$, $1 = 1, 2$;

$\mathcal{M}, m \Vdash \phi_1 - \phi_2$ iff $[m'' \equiv m \otimes m'$ and $\mathcal{M}, m' \Vdash \phi_2]$ implies $\mathcal{M}, m'' \Vdash \phi_1$.

"|" and " − " are polyadic modalities of arity 2. " − " is the adjoint of "|". The formula $L_r^a \phi$ is interpreted as *"the rate of an a-transition from the current state to a state satisfying* ϕ *is at least r"*. Notice that the semantics of $L_r^a \phi$ is well defined only if $[\![\phi]\!]_{\mathcal{M}}$ is measurable. This is guaranteed by the next lemma.

Lemma 2. *For any* $\phi \in \mathcal{L}$ *and any* $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$*,* $[\![\phi]\!]_{\mathcal{M}} \in \Sigma$.

When it is not the case that $\mathcal{M}, m \Vdash \phi$, we write $\mathcal{M}, m \nVdash \phi$. A formula ϕ is *satisfiable* if there exists $\mathcal{M} \in \mathfrak{M}$ and $m \in sup(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$. If $\neg\phi$ is not satisfiable, ϕ is *valid*, denoted by $\Vdash \phi$.

Let $\boxed{+}_{i=1..n} \phi_i = \bigwedge_{i,j=1..n}^{i \neq j} (\phi_i \to \neg\phi_j)$ and $\bot = \neg\top$. For each $k \in \mathbb{N}$, let $k = \neg(\underbrace{\top|\top|..|\top}_{k+1})$; $\mathcal{M}, m \Vdash k$ iff m can be decomposed in maximum k modules.

Regarding the expresivity of MML, there are some interesting derived operators that can be defined.

$\phi_1 \parallel \phi_2 = \neg(\neg\phi_1|\neg\phi_2)$ is the De Morgan dual of "|"; $\mathcal{M}, m \Vdash \phi_1 \parallel \phi_2$ iff $m \equiv m_1 \otimes m_2$ implies $\mathcal{M}, m_i \Vdash \phi_1$ or $\mathcal{M}, m_j \Vdash \phi_2$, $\{i, j\} = \{1, 2\}$.

$\phi_1 * \phi_2 = \neg(\neg\phi_1 - \neg\phi_2)$ is the De Morgan dual of "−"; $\mathcal{M}, m \Vdash \phi_1 * \phi_2$ iff there exists $\mathcal{M}, n \nVdash \phi_2$ and $m' \equiv m \otimes n$ such that $\mathcal{M}, m' \nVdash \phi_1$.

Let $\circ\phi = \bot - (\neg\phi)$. Notice that it encodes the validity of ϕ in a MMK, $\mathcal{M}, m \Vdash \circ\phi$ iff for any $n \in sup(M)$, $\mathcal{M}, n \Vdash \phi$.

Similarly, the De Morgan dual of "$\circ$" encodes the satisfiability in a MMK: $\bullet\phi = \neg \circ \neg\phi$, $\mathcal{M}, m \Vdash \bullet\phi$ iff there exists $n \in M$, $\mathcal{M}, n \Vdash \phi$.

In the rest of this section we focus on the logical equivalence induced by MML on MMPs and its relation to stochastic bisimulation on MMPs. The next theorem states that $\equiv$ preserves the satisfiability of $\mathcal{L}$ formulas.

Theorem 2. *For $\mathcal{M} \in \mathfrak{M}$ and $m, n \in \mathrm{sup}(\mathcal{M})$, if $m \equiv n$, then for all $\phi \in \mathcal{L}$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}, n \Vdash \phi$.*

Let $\mathcal{L}^* \subsetneq \mathcal{L}$ be defined by the grammar $\phi := \top \vdots \neg\phi \vdots \phi \wedge \phi \vdots L_r^a\phi$. The next theorem reproduces a similar result presented in [14, 29].

Theorem 3. *Given $\mathcal{M} \in \mathfrak{M}$ and $m, n \in \mathrm{sup}(\mathcal{M})$, if [for any $\phi \in \mathcal{L}^*$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}, n \Vdash \phi$], then $m \sim n$.*

5 A complete Hilbert-style axiomatization for MML

Tables 1, 2 and 3 contain a Hilbert-style axiomatization for MML.

The stochastic axioms in Table 1 have been proposed in [10] where we have proved that they form a complete axiomatization for CMPs. These axioms are similar, but more complex due to stochasticity, than the ones proposed in [32] for Harsanyi type spaces. As in the probabilistic case, we have infinitary rules (R2) and (R3) that encode the Archimedean properties of $\mathbb{Q}$. However, a finitary axiomatization is possible on the lines of [19] at the price of defining some complex operators, as shown in [23].

Table 1: Stochastic Axioms

(A1):	$\vdash L_0^a\phi$
(A2):	$\vdash L_{r+s}^a\phi \to L_r^a\phi$
(A3):	$\vdash L_r^a(\phi \wedge \psi) \wedge L_s^a(\phi \wedge \neg\psi) \to L_{r+s}^a\phi$
(A4):	$\vdash \neg L_r^a(\phi \wedge \psi) \wedge \neg L_s^a(\phi \wedge \neg\psi) \to \neg L_{r+s}^a\phi$
(R1):	If $\vdash \phi \to \psi$ then $\vdash L_r^a\phi \to L_r^a\psi$
(R2):	If $\forall r < s, \vdash \phi \to L_r^a\psi$ then $\vdash \phi \to L_s^a\psi$
(R3):	If $\forall r > s, \vdash \phi \to L_r^a\psi$ then $\vdash \phi \to \bot$

Table 2: Structural Axioms

(A5):	$\vdash (\phi|\psi)|\rho \to \phi|(\psi|\rho)$
(A6):	$\vdash \phi|\psi \to \psi|\phi$
(A7):	$\vdash \phi|\bot \to \bot$
(A8):	$\vdash \phi|(\psi \vee \rho) \to (\phi|\psi \vee \phi|\rho)$
(R4):	If $\vdash \phi \to \psi$ then $\vdash \phi|\rho \to \psi|\rho$
(R5):	If $\vdash \phi \to \phi|\top$ then $\vdash \phi \to \bot$
(R6):	$\vdash \phi \to (\psi - \rho)$ iff $\vdash \phi|\rho \to \psi$

The structural axioms in Table 2 are similar to the axioms proposed in [26] for a spatial logic on CCS semantics. The main difference is rule (R5) which rejects models that do not respect the modularity conditions. An example is the rule (Nill): $P \equiv P|0$ which allows processes with (trivial) non-wellfounded structure. However, one can easily make an MMP from a process algebra term by simply taking the quotient of the class of processes by (Nill) and similar rules.

To introduce the modular axioms in Table 3 we fix some notations. π_k is the set of permutations of $\{1,..,k\}$. For $a \in \mathcal{A}$, let $a^* = \{(b_i, c_i) \mid i \in I_a\}$. If $\{(r_k^{i,j}) \mid i \in I, k \in K, j \in \{0,1\}\}, \{(s_k^{i,j}) \mid i \in I, k \in K, j \in \{0,1\}\} \subseteq \mathbb{Q}_+$, let $r_K^I \bullet s_K^I = \sum_{i\in I} \sum_{j\in\{0,1\}}^{k\in K} (r_k^{i,j}) \bullet (s_k^{i,j})$.

Table 3: Modular axioms

(R7): If I is finite and $\vdash 1 \to \bigvee_{i\in I} \phi_i$, then $\vdash k \to \bigvee_{i_j\in I}^{s<k} \phi_{i_1}|...|\phi_{i_s}$.

(R8): If $\vdash \bigwedge_{i=1..k}^{j=0,1} (\phi_i^j \to 1)$ and $\vdash \phi_1^0|..|\phi_k^0 \to \phi_1^1|..|\phi_l^1$, then $k = l$ and

$$\vdash \bigvee_{\sigma\in\pi_k} \bigwedge_{i=1..k} \phi_i^0 \leftrightarrow \phi_{\sigma(i)}^1$$

(R9): If K is finite and $\vdash (\boxed{+}_{k\in K} \phi_k^0) \wedge (\boxed{+}_{k\in K} \phi_k^1) \wedge (\bigvee_{k\in K} \phi_k^0|\phi_k^1 \to \rho)$, then

$$\vdash \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} L_{(r_k^{i,j})}^{b_i} \phi_k^j\right) \Big| \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} L_{(s_k^{i,j})}^{c_i} \phi_k^{1-j}\right) \to L_{(r_K^I \bullet s_K^I)}^a \rho$$

(R10): If K is finite and $\vdash (\boxed{+}_{k\in K} \phi_k^0) \wedge (\boxed{+}_{k\in K} \phi_k^1) \wedge (\rho \to \bigvee_{k\in K} \phi_k^0|\phi_k^1)$, then

$$\vdash \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} \neg L_{(r_k^{i,j})}^{b_i} \phi_k^j\right) \Big| \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} \neg L_{(s_k^{i,j})}^{c_i} \phi_k^{1-j}\right) \to \neg L_{(r_K^I \bullet s_K^I)}^a \rho$$

The rules (R7) and (R8) encode the wellfoundness and the modularity of the models. The rules (R9) and (R10) are logical versions of classical *expansion laws* for parallel operator. (R9) states that the rate of the a-transitions from m to $[\![\rho]\!]$ is at least the sum, after $k \in K$, of all $\bullet$-products of the rates of b and c-transitions (for $a = b * c$) from m_1 and m_2 to $[\![\phi_k^j]\!]$ and $[\![\phi_k^{1-j}]\!]$ respectively ($j = 0, 1$), given that $m \equiv m_1 \otimes m_2$ and $[\![\rho]\!]$ covers $\bigcup_{k\in K} [\![\phi_k^j]\!] \otimes [\![\phi_k^{1-j}]\!]$. For instance, $\vdash (L_r^b\phi \wedge L_u^c\psi)|(L_s^c\psi \wedge L_v^b\phi) \to L_{(r\bullet s)+(u\bullet v)}^{b*c}\phi|\psi$ and $\vdash L_r^b\top|L_s^c\top \to L_{r\bullet s}^{b*c}\top$ are instances of (R9). Similarly, (R10) states that the rate of the a-transitions from m to $[\![\rho]\!]$ is strictly bigger than the sum, after $k \in K$, of all $\bullet$-products of the rates of b and c-transitions (for $a = b * c$) from m_1 and m_2 to $[\![\phi_k^j]\!]$ and $[\![\phi_k^{1-j}]\!]$ respectively ($j = 0, 1$), given that $m \equiv m_1 \otimes m_2$ and $[\![\rho]\!]$ is covered by $\bigcup_{k\in K} [\![\phi_k^j]\!] \otimes [\![\phi_k^{1-j}]\!]$. $\vdash (\neg L_r^b\top \wedge \neg L_u^c\top)|(\neg L_s^c\top \wedge \neg L_v^b\top) \to \neg L_{(r\bullet s)+(u\bullet v)}^a(\top|\top)$ is an instance of (R10) given that b, c are the only actions such that $a = b * c$.

We say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the previous axioms (using also Boolean rules). We say that ϕ is *consistent*, if $\phi \to \bot$ is not provable. Given a set $\Phi, \Psi \subseteq \mathcal{L}$, Φ proves Ψ if from the formulas of Φ and the axioms we can prove all $\psi \in \Psi$; we write $\Phi \vdash \Psi$. Φ is consistent if it is not the case that $\Phi \vdash \bot$. For a sublanguage $L \subseteq \mathcal{L}$, we call Φ *L-maximally consistent* if Φ is consistent and no formula of L can be added to it without making it inconsistent.

Theorem 4 (Soundness). *The axiomatic system of MML is sound for the Markovian semantics, i.e., for any $\phi \in \mathcal{L}$, if $\vdash \phi$ then $\Vdash \phi$.*

In what follows we prove the finite model property for MML by constructing a model for a given consistent formula. This result will eventually prove that the axiomatic system is also complete for the Markovian semantics, meaning that everything that is true for all the models can be proved. Before proceeding, we fix some notations.

For $n \in \mathbb{N}$, $n \neq 0$, let $\mathbb{Q}_n = \{\frac{p}{n} : p \in \mathbb{N}\}$. If $S \subseteq \mathbb{Q}$ is finite, the *granularity of* S, $gr(S)$, is the lowest common denominator of the elements of S.
The *modal depth of* $\phi \in \mathcal{L}$ is defined by $md(\top) = 0$, $md(\neg\phi) = md(\phi)$, $md(L_r^a\phi) = md(\phi) + 1$ and $md(\phi \wedge \psi) = md(\phi|\psi) = md(\phi - \psi) = max(md(\phi), md(\psi))$.
The *structural depth of* $\phi \in \mathcal{L}$ is defined by $sd(\neg\phi) = sd(L_r^a\phi) = sd(\phi)$, $sd(\phi \wedge \psi) = max(sd(\phi), sd(\psi))$ and $sd(\phi|\psi) = sd(\phi - \psi) = sd(\phi) + sd(\psi) + 1$.
The *granularity of* $\phi \in \mathcal{L}$ is $gr(\phi) = gr(R)$, where $R \subseteq \mathbb{Q}_+$ is the set of indexes r of the operators L_r^a present in ϕ; the *upper bound of* ϕ is $max(\phi) = max(R)$.
For $\Lambda_1, \Lambda_2 \subseteq \mathcal{L}$, $\Lambda_1|\Lambda_2 = \{\phi_1|\phi_2 : \phi_i \in \Lambda_i, i = 1, 2\}$. ϕ is a *generator* for Λ_1 if $\phi \vdash \Lambda_1$.
For arbitrary $n \in \mathbb{N}$, let $\mathcal{L}_n$ be the sublanguage of $\mathcal{L}$ that uses only modal operators L_r^a with $r \in \mathbb{Q}_n$ and $\mathcal{L}_n^k = \{\psi \in \mathcal{L}_n \mid sd(\psi) \leq k\}$. For $\Lambda \subseteq \mathcal{L}$, let $[\Lambda]_n = \{\phi \in \mathcal{L}_n : \Lambda \vdash \phi\}$ and $[\Lambda]_n^k = \{\phi \in \mathcal{L}_n^k : \Lambda \vdash \phi\}$.

Consider a consistent formula $\psi \in \mathcal{L}$ with $gr(\psi) = n$ and $sd(\psi) = e$.
Let $\mathcal{L}[\psi] = \{\phi \in \mathcal{L}_n^e \mid max(\phi) \leq max(\psi), md(\phi) \leq md(\psi)\}$ and $\mathcal{L}^0[\psi] = \mathcal{L}[\psi] \cap \mathcal{L}_n^0$.

In what follows we construct $\mathcal{M}_\psi \in \mathfrak{M}$ such that each $\Gamma \in \text{sup}(\mathcal{M}_\psi)$ is a consistent set of formulas that contains an $\mathcal{L}[\psi]$-maximally consistent set and each $\mathcal{L}[\psi]$-maximally consistent set is contained in some $\Gamma \in \text{sup}(\mathcal{M}_\psi)$. And we will prove that for $\phi \in \mathcal{L}[\psi]$, $\phi \in \Gamma$ iff $\mathcal{M}_\psi, \Gamma \Vdash \phi$.

Let $\Omega^0[\psi]$ be the set of $\mathcal{L}^0[\psi]$-maximally consistent sets of formulas. $\Omega^0[\psi]$ is finite and any $\Lambda \in \Omega^0[\psi]$ contains finitely many nontrivial formulas[3]; in the rest of this construction we only count non-trivial formulas while ignoring the rest.

For each $\Lambda \in \Omega[\psi]$, such that $\{\phi_1, ..., \phi_i\}$ is the set of its non-trivial formulas, we construct $\Lambda^+ \supseteq [\Lambda]_n^0$ with the property that $\forall \phi \in \Lambda$ and $a \in \mathcal{A}$ there exists $\neg L_r^a\phi \in \Lambda^+$.
The step [ϕ_1 and Λ:] (R3) guarantees that $\exists r \in \mathbb{Q}_n$ s.t. $[\Lambda]_n^0 \cup \{\neg L_r^a\phi_1\}$ is consistent. Let $y_1^a = min\{s \in \mathbb{Q}_n : [\Lambda]_n^0 \cup \{\neg L_s^a\phi_1\}$ is consistent$\}$ and $x_1^a = max\{s \in \mathbb{Q}_n : L_s^a\phi_1 \in [\Lambda]_n^0\}$ ((R3) guarantees the existence of max). (R2) implies that $\exists r \in \mathbb{Q} \setminus \mathbb{Q}_n$ s.t., $x_1^a < r < y_1^a$ and $\{\neg L_r^a\phi_1\} \cup [\Lambda]_n^0$ is consistent. Let $n_1 = gran\{1/n, r\}$. Let $s_1^a = min\{s \in \mathbb{Q}_{n_1} : [\Lambda]_{n_1}^0 \cup \{\neg L_s^a\phi_1\}$ is consistent$\}$, $\Lambda_1^a = \Lambda \cup \{\neg L_{s_1}^a\phi_1\}$ and $\Lambda_1 = \bigcup_{a \in A} \Lambda_1^a$.

We repeat this step of the construction for [ϕ_2 and Λ_1],..,[ϕ_i and Λ_{i-1}] and we obtain $\Lambda \subseteq \Lambda_1 \subseteq ... \subseteq \Lambda_i$, where Λ_i is a consistent set containing a finite set of nontrivial formulas. Let $n_\Lambda = gran\{1/n_1, .., 1/n_i\}$. We make this construction for all $\Lambda \in \Omega^0[\psi]$. Let $p = gran\{1/n_\Lambda : \Lambda \in \Omega^0[\psi]\}$. Notice that $p > n$. Let $\Lambda^+ = [\Lambda_i]_p \cup \{1\}$ (this is consistent) and $\Omega^+[\psi] = \{\Lambda^+ : \Lambda \in \Omega^0[\psi]\}$. The construction is finite.

Remark 1. For each $\Lambda \in \Omega^0[\psi]$, $\phi \in \Lambda$ and $a \in \mathcal{A}$, there exist $s, t \in \mathbb{Q}_p$, $s < t$, such that $L_s^a\phi, \neg L_r^a\phi \in \Gamma^+$. Moreover, there exists a generator $f \in \Lambda^+$ of Λ^+. For each Λ we fix such a generator and let $\mathcal{F}$ be the set of all generators of the elements of $\Omega^+[\psi]$.

[3] By nontrivial formulas we mean the formulas that are not obtained from more basic consistent ones by boolean derivations.

(R5) guarantees that for any $\phi \in \mathcal{L}$ there exists $k \in \mathbb{N}$ such that $\vdash \phi \to k$. Let $\overline{k} = min_{\phi \in \mathcal{L}^0[\psi]}\{k, \vdash \phi \to k\}$. Let $\Omega[\psi] = \{[\Lambda^+_{i_1}|...|\Lambda^+_{i_l}]_p$ for $\Lambda^+_{i_j} \in \Omega^0[\psi]$, $j = 1..l, l \leq \overline{k}\}$.

Lemma 3. *Any consistent formula $\phi \in \mathcal{L}[\psi]$ is contained in at least one $\Pi \in \Omega[\psi]$. For arbitrary $\Pi_1, \Pi_2 \in \Omega[\psi]$, there exists a unique $\Pi \in \Omega[\psi]$ such that $\Pi_1|\Pi_2 \subseteq \Pi$. $\mathcal{F}^* = \{f_{i_1}|...|f_{i_l}$ where $f_{i_j} \in \mathcal{F}$, $j = 1..l, l \leq \overline{k}\}$ is a set of generators for all $\Pi \in \Omega[\psi]$.*

Let Ω_p be the set of $\mathcal{L}_p$-maximally consistent sets of formulas and $\sigma : \Omega[\psi] \to \Omega_p$ be an injection such that for any $\Pi \in \Omega[\psi]$, $\Pi \subseteq \sigma(\Pi)$. We denote by $\Omega_p[\psi] = \sigma(\Omega[\psi])$. For $\phi \in \mathcal{L}[\psi]$, let $[\![\phi]\!] = \{\Gamma \in \Omega_p[\psi] : \phi \in \Gamma\}$.

Lemma 4. *(1) $\Omega_p[\psi]$ is finite.* *(2) $2^{\Omega_p[\psi]} = \{[\![\phi]\!] \mid \phi \in \mathcal{L}[\psi]\}$.*
(3) For any $\phi_1, \phi_2 \in \mathcal{L}[\psi]$, $\vdash \phi_1 \to \phi_2$ iff $[\![\phi_1]\!] \subseteq [\![\phi_2]\!]$.
(4) For any $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}[\psi]$ and $a \in \mathcal{A}$, there exist
$x = max\{r \in \mathbb{Q}_p : L^a_r\phi \in \Gamma\}$, $y = min\{r \in \mathbb{Q}_p : \neg L^a_r\phi \in \Gamma\}$ and $y = x + 1/p$.

$\Omega_p[\psi]$ will be the support of $\mathcal{M}_\psi$. It has the property that for each consistent $\phi \in \mathcal{L}[\psi]$ there exists $\Gamma \in \Omega_p[\psi]$ such that $\phi \in \Gamma$ and for each Γ and $a \in \mathcal{A}$, there exists $r \in \mathbb{Q}_p$ such that $L^a_r\phi, \neg L^a_{r+1/p}\phi \in \Gamma$. This is still not sufficient to define θ_ψ.

Let Ω be the set of $\mathcal{L}$-maximally consistent sets of formulas and $\pi : \Omega_p \to \Omega$ an injection such that for any $\Gamma \in \Omega_p$, $\Gamma \subseteq \pi(\Gamma)$; let $\Gamma^\infty = \pi(\Gamma)$.

Lemma 5. *For any $\Gamma \in \Omega_p[\psi]$, any $\phi \in \mathcal{L}[\psi]$ and any $a \in \mathcal{A}$, there exist*
$x^\infty = sup\{r \in \mathbb{Q} : L^a_r\phi \in \Gamma^\infty\} = inf\{r \in \mathbb{Q} : \neg L^a_r\phi \in \Gamma^\infty\}$ and $x \leq x^\infty < y$.

We denote by $a^\Gamma_\phi = x^\infty$ defined for $\phi \in \mathcal{L}[\psi]$, $\Gamma \in \Omega_q[\psi]$ and $a \in \mathcal{A}$.

Lemma 6. *$\mathcal{M}_\psi = (\mathcal{K}_\psi, =, \otimes) \in \mathfrak{M}$, where $\mathcal{K}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi)$ with θ_ψ defined for arbitrary $\phi \in \mathcal{L}[\psi]$, $\Gamma \in \Omega_p[\psi]$, $a \in \mathcal{A}$ by $\theta_\psi(a)(\Gamma)([\![\phi]\!]) = a^\Gamma_\phi$, and $\otimes$ defined for arbitrary $\Gamma, \Gamma', \Gamma'' \in \Omega_p[\psi]$ by $[\Gamma = \Gamma' \otimes \Gamma''$ iff $\Gamma'|\Gamma'' \subseteq \Gamma]$.*

Now we can prove the Truth Lemma.

Lemma 7 (Truth Lemma). *If $\phi \in \mathcal{L}[\psi]$ and $\Gamma \in \Omega_p[\psi]$, then $[\mathcal{M}_\psi, \Gamma \Vdash \phi$ iff $\phi \in \Gamma]$.*

Proof. Induction on the structure of ϕ. The Boolean cases are trivial.
The case $\phi = L^a_r\phi'$**:** ($\Longrightarrow$) Suppose that $\mathcal{M}_\psi, \Gamma \Vdash L^a_r\phi'$ and $L^a_r\phi' \notin \Gamma$. Because Γ is $\mathcal{L}[\psi]$-maximally consistent, $\neg L^a_r\phi' \in \Gamma$. Let $y = min\{r \in \mathbb{Q}_p : \neg L^a_r\phi' \in \Gamma\}$. Then, from $\neg L^a_r\phi' \in \Gamma$, we obtain $r \geq y$. But $\mathcal{M}_\psi, \Gamma \Vdash L^a_r\phi'$ is equivalent with $\theta_\psi(a)(\Gamma)([\![\phi']\!]) \geq r$, i.e. $a^\Gamma_{\phi'} \geq r$. On the other hand, in Lemma 4 we proved that $a^\Gamma_{\phi'} < y$ - contradiction.
($\Longleftarrow$) Suppose that $L^a_r\phi' \in \Gamma$. Then $r \leq a^\Gamma_\phi$, implying $\theta_\psi(a)(\Gamma)([\![\phi]\!]) \geq r$.
The case $\phi = \phi_1|\phi_2$**:** ($\Longrightarrow$) If $\mathcal{M}_\psi, \Gamma \Vdash \phi_1|\phi_2$, then $\Gamma = \Gamma_1 \otimes \Gamma_2$ and $\mathcal{M}_\psi, \Gamma_i \Vdash \phi_i$, $i = 1, 2$. The inductive hypothesis implies that $\phi_i \in \Gamma_i$ and because $\Gamma_1|\Gamma_2 \subseteq \Gamma$, $\phi_1|\phi_2 \in \Gamma$.
($\Longleftarrow$) If $\phi_1|\phi_2 \in \Gamma$, then Axiom (A7) guarantees that there exists Γ_i with $\phi_i \in \Gamma_i$ and $\Gamma_1|\Gamma_2 \subseteq \Gamma$, i.e. $\Gamma = \Gamma_1 \otimes \Gamma_2$. The inductive hypothesis gives $\mathcal{M}_\psi, \Gamma_i \Vdash \phi_i$.
The case $\phi = \phi_1 - \phi_2$**:** ($\Longrightarrow$) Suppose that $\mathcal{M}_\psi, \Gamma \Vdash \phi_1 - \phi_2$. Then, $\mathcal{M}_\psi, \Gamma_2 \Vdash \phi_2$ implies $\mathcal{M}_\psi, \Gamma \otimes \Gamma_2 \Vdash \phi_1$. Using the inductive hypothesis, $\phi_2 \in \Gamma_2$ implies $\phi_1 \in \Gamma|\Gamma_2$, i.e. $\vdash f|\phi_2 \to \phi_1$, where f is the generators of Γ. Further, (R4) proves $\vdash f \to (\phi_1 - \phi_2)$.
($\Longleftarrow$) $\phi_2 - \phi_1 \in \Gamma$ implies $\vdash f \to (\phi_2 - \phi_1)$ and from Rule (R4), $\vdash f|\phi_2 \to \phi_1$. Hence, $\Gamma_2 \in [\![\phi_2]\!]$ implies $\phi_1 \in \Gamma \otimes \Gamma_2$. The inductive hypothesis ends the proof.

The previous lemma implies the small model property for our logic.

Theorem 5 (Small model property). *For any consistent formula ϕ, there exists $\mathcal{M} \in \mathfrak{M}$ with the cardinality of $sup(\mathcal{M})$ bound by the structure of ϕ, and $m \in sup(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$.*

The small model property proves the completeness of the axiomatic system.

Theorem 6 (Completeness). *MML is complete with respect to the Markovian semantics, i.e. if $\Vdash \psi$, then $\vdash \psi$.*

Proof. We have shown that any consistent formula has a model. We prove that this is sufficient for completeness. Indeed, [$\Vdash \psi$ implies $\vdash \psi$] is equivalent with [$\nvdash \psi$ implies $\nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies that there exists a model $\mathcal{M}$ such that $\mathcal{M}, m \nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies the satisfiability of $\neg\psi$]. This last term is equivalent to our working hypothesis.

6 Conclusions and future work

In this paper we introduce Modular Markovian Logic, a new logic that combines features of stochastic and modular logics. Its semantics is in terms of modular Markov processes which are compositional continuous-time and continuous-space Markov processes. MML is appropriate for specifying and verifying modular properties of stochastic systems and to prove global properties from local properties of subsystems. For instance modular proof rules as the ones bellow can be given as instances of (R9).

$$\frac{P \Vdash L_r^b\top, \;\; P' \Vdash L_s^c\top}{P\otimes P' \Vdash L_{r\bullet s}^{b*c}\top} \quad \text{and} \quad \frac{P \Vdash L_r^b\phi \wedge L_u^c\psi, \;\; P' \Vdash L_s^c\psi \wedge L_v^b\phi}{P\otimes P' \Vdash L_{(r\bullet s)+(u\bullet v)}^{b*c}\rho} \; (\vdash \phi|\psi \rightarrow \rho).$$

Similarly, if b, c are unique such that $a = b*c$ and P, P' are unique such that $P'' \equiv P\otimes P'$, the rule bellow is based on an instance of (R10).

$$\frac{P \Vdash \neg L_r^b\top \wedge \neg L_u^c\top, \quad P' \Vdash \neg L_s^c\top \wedge \neg L_v^b\top}{P'' \Vdash \neg L_{(r\bullet s)+(u\bullet v)}^a(\top|\top)}$$

In this paper we present a complete Hilbert-style axiomatization for MML and prove the small model property. For future work we intend to focus on decidability and complexity problems as well as on axiomatizations of model checking and possible procedures to automatize the proof of modular rules.

References

1. R. Aumann, *Interactive epistemology: Probability.* Int. J. Game Theory, 28, 1999.
2. J.A. Bergstra, J. W. Klop, *ACP: A Universal Axiom System for Process Specification.* CWI Quarterly 15:3-23, 1987.
3. J.A. Bergstra et.al (Eds.), *Handbook of Process Algebra.* Elsevier, 2001.
4. R. Blute, J. Desharnais, A. Edalat, P. Panangaden. *Bisimulation for labeled Markov processes.* In Proc. of LICS'97, IEEE Press, 1997.

5. M. Bravetti, H. Hermanns, J.-P. Katoen. *YMCA: Why Markov Chain Algebra?* ENTCS 162, 2006
6. L. Caires, L. Cardelli, *A Spatial Logic for Concurrency.* Inf. and Comp. 186(2), 2003.
7. L. Cardelli, A. Gordon, *Anytime, anywhere: Modal logics for mobile ambients.* In Proc. of POPL00, pages 365377, 2000.
8. L. Cardelli, R. Mardare, *The Measurable Space of Stochastic Processes.* QEST 2010, IEEE Press, 2010.
9. L. Cardelli, R. Mardare, *Stochastic-pi Calculus revisited.* unpublished manuscript available at http://lucacardelli.name/.
10. L. Cardelli, R. Mardare, *Continuous Markovian Logic.* Tech.Rep. TR-4-2010, The Microsoft Research-CoSBi Centre, 2010.
11. E. Doberkat, *Stochastic Relations. Foundations for Markov Transition Systems*. Chapman and Hall/CRC, 2007.
12. J. Desharnais, A. Edalat, P. Panangaden, *A logical characterization of bisimulation for labeled Markov processes*, In Proc of LICS'98, IEEE Press, 1998.
13. J. Desharnais, A. Edalat, P. Panangaden, *Bisimulation for labeled Markov Processes*, Inf. and Comp., 179(2):163-193, 2002.
14. J. Desharnais, P. Panangaden, *Continous Stochastic Logic Characterizes Bisimulation of Continuous-time Markov Processes*, JLAP. 56:99-115, 2003.
15. E.A. Emerson, *Temporal and modal logic.* Handbook of Theoretical Computer Science, MIT Press, 1990.
16. R. Fagin, J. Halpern, *Reasoning about knowledge and probability.* J. ACM, 41:340-367, 1994.
17. J. Harsanyi, *Games with incomplete information played by bayesian players, part one.* Management Sci., 14:159-182, 1967.
18. J. Hillston, *Process algebras for quantitative analysis.* In Proc. LICS'05, IEEE Press, 2005.
19. A. Heifetz, P. Mongin, *Probability Logic for Type Spaces.* Games and Economic Behavior 35, 3153, 2001.
20. M. Hennessy, R. Milner, *On observing nondeterminism and concurrency.* LNCS 85/1980, 299-309, 1980.
21. C. A. R. Hoare, *Communicating sequential processes.* Communications of the ACM 21(8):666677, 1978.
22. D. Kozen, *Results on the Propositional Mu-Calculus.* TCS 27(3):333354, 1983.
23. C. Kupke, D. Pattinson, *On Modal Logics of Linear Inequalities.* In Proc of AiML 2010.
24. M. Kwiatkowska et al., *Automatic Verification of Real-Time Systems With Discrete Probability Distributions.*, LNCS 1601:79-95, 1999.
25. K.G. Larsen and A. Skou, *Bisimulation through probabilistic testing.* Inf. and Comp., 94:1-28, 1991.
26. R. Mardare, A. Policriti, *A Complete Axiomatic System for Process-based Spatial Logic.* In Proc. of MFCS 2008, LNCS 5168:491-502, 2008.
27. R. Milner, *A calculus of communicating systems.* Journal of the ACM, 1980.
28. L.S. Moss, I.D. Viglizzo, *Harsanyi type spaces and final coalgebras constructed from satisfied theories.* ENTCS 106:279-295, 2004.
29. P. Panangaden, *Labelled Markov Processes.* Imperial College Press, 2009.
30. C. Priami, *Stochastic π-Calculus.* Computer Journal, 38(7):578-589, 1995.
31. J. C. Reynolds, *Separation Logic: A Logic for Shared Mutable Data Structures.* In Proc. of LICS'02, IEEE Press, 2002.
32. C. Zhou, *A complete deductive system for probability logic with application to Harsanyi type spaces*, PhD thesis, Indiana University, 2007.

Appendix

In this appendix we have collected the proofs of the major results presented in the paper.

Proof (Proof of Lemma 1). We have to prove that $\theta : \mathcal{A} \to [\![M \to \Delta(M, \Sigma)]\!]$.

We prove first that for each $a \in \mathcal{A}$ and each $(m, n) \in M_1 \times M_2$, $\theta(a)((m, n))$ is a measure. It is trivial to check that $\theta(a)((m, n))(\emptyset) = 0$.

Suppose that $S = \bigcup_{i \in I} S^i$, where $S^i \in \Sigma$, $S^i \cap S^j = \emptyset$ for $i \neq j$. We prove that $\theta(a)((m, n))(S) = \sum_{i \in I} \theta(a)((m, n))(S^i)$.

Let Δ_i be the denumerable bases of Σ_i, $i = 1, 2$. Then,

$$S^i = \bigcup_{j \in J_i} U^i_j \times V^i_j \text{ and } S = \bigcup_{i \in I} \bigcup_{j \in J_i} U^i_j \times V^i_j,$$

where $U^i_j \in \Delta_1$, $V^i_j \in \Delta_2$. Now, because $\bullet$ is distributive, we obtain

$$\theta(a)((m, n))(S) = \sum_{b*c=a} \sum_{i \in I} \sum_{j \in J_i} \theta_1(b)(m)(U^i_j) \bullet \theta_2(c)(n)(V^i_j) =$$

$$\sum_{i \in I} \sum_{b*c=a} \sum_{j \in J_i} \theta_1(b)(m)(U^i_j) \bullet \theta_2(c)(n)(V^i_j)] = \sum_{i \in I} \theta((m, n))(\tau)(S^i).$$

This proves that $\theta(a)((m, n)) \in \Delta(M, \Sigma)$.

Because $\theta(a)((m, n))$ is a measure, we also obtain that the sum is convergent, being monotonic and bound.

It remains to prove that for any $a \in \mathcal{A}$, $\theta(a) : M \to \Delta(M, \Sigma)$ is a measurable function. For this, it is sufficient to show that for fixed $a \in \mathcal{A}$, $S \in \Sigma$ and $r \in \mathbb{Q}_+$, $\{(m, n) \in M \mid \theta(a)((m, n))(S) \leq r\} \in \Sigma$. Assume that $S = \bigcup_{i \in I} U_i \times V_i$ with $U_i \in \Delta_1$ and $V_i \in \Delta_2$.

The most general nontrivial case is to assume that there exist $k, l \in I$ such that $m \in U_k$ and $n \in V_l$. Then,

$$\theta(a)((m, n))(S) = \sum_{b*c=a} \theta_1(b)(m)(U_l) \bullet \theta_2(c)(n)(V_k).$$

Hence, $\{(m, n) \in M \mid \theta(a)((m, n))(S) \leq r\} =$

$$\bigcup_{\sum_{b*c=a} (s_b \bullet s_c) \leq r}^{\{s_b \mid b \in \mathcal{A}\} \subseteq \mathbb{Q}_+} \{m \in M_1 \mid \theta_1(b)(m)(U_l) \leq s_b\} \times \{n \in M_2 \mid \theta_2(c)(n)(V_k) \leq s_c\}].$$

and because $\theta_i(a)$ are measurable for $i = 1, 2$, we obtain the desired result.

Proof (Theorem 1). We define the relation $\approx$ as follows. If $m_1 \sim_{\mathcal{K}} m_2$ and $m_1 \otimes n, m_2 \otimes n$ are defined, then $(m_1, n) \approx (m_2, n)$. It is sufficient to prove that $\approx$ is a rate bisimulation.

Suppose that $\mathcal{K} = (M, \Sigma, \theta)$. Let $\Theta = \overline{\Sigma \times \Sigma}$ and consider an arbitrary $C \in \Theta(\approx)$. If we denote by ρ the transition function of $\mathcal{K} \times \mathcal{K}$, we need to prove that $\rho(a)((m_1, n))(C) =$

$\rho(a)((m_2,n))(C)$ for any $a \in \mathcal{A}$. Actually, it is sufficient to prove this for arbitrary $C = S_1 \times S_2$, where $S_1 \in \Sigma(\sim)$ and $S_2 \in \Sigma$. We have

$$\rho(a)((m_i,n))(S_1 \times S_2) = \sum_{b*c=a} \theta(b)(m_i)(S_1) \bullet \theta(c)(n)(S_2).$$

Because $m_1 \sim_{\mathcal{K}} m_2$ and $S_1 \in \Sigma(\sim)$, for each $b \in \mathcal{A}$, $\theta(b)(m_1)(S_1) = \theta(b)(m_2)(S_1)$. Consequently, $\approx$ is a rate bisimulation.

Proof (Lemma 2). Induction on the structure of ϕ.
The case $\phi = L^a_r\psi$**:** $[\![L^a_r\psi]\!]_{\mathcal{M}} = (\theta(a))^{-1}(\{\mu \in \Delta(M,\Sigma)|\mu([\![\psi]\!]_{\mathcal{M}}) \geq r\})$. From the inductive hypothesis, $[\![\psi]\!]_{\mathcal{M}} \in \Sigma$, hence, $\{\mu \in \Delta(M,\Sigma)|\mu([\![\psi]\!]_{\mathcal{M}}) \geq r\}$ is measurable in $\Delta(M,\Sigma)$ and because $\theta(a)$ is a measurable mapping, we obtain that $[\![L^a_r\psi]\!]_{\mathcal{M}}$ is measurable.
The cases $\phi = \phi_1|\phi_2$**:** $[\![\phi[\![= [\![\phi_1]\!]\otimes[\![\phi_2]\!]$ and because $[\![\phi_1]\!], [\![\phi_2]\!] \in \Sigma$ and $\Sigma\otimes\Sigma \subseteq \Sigma$, we get $[\![\phi]\!] \in \Sigma$.
The cases $\phi = \phi_1 - \phi_2$**:** $[\![\phi]\!] = [\![\phi_1]\!]\otimes^-[\![\phi_2]\!]$ and because $[\![\phi_1]\!], [\![\phi_2]\!] \in \Sigma$ and $\Sigma\otimes^-\Sigma \subseteq \Sigma$, we get $[\![\phi]\!] \in \Sigma$.

Proof (Theorem 2). Induction on the structure of $\phi \in \mathcal{L}$. The Boolean cases are trivial.

The case $\phi = L^a_r\psi$**:** $m \equiv n$ implies $m \sim n$ and consequently $\theta(a)(m)(C) = \theta(a)(n)(C)$ for any $C \in \Sigma(\sim)$. But from the inductive hypothesis we have that $[\![\psi]\!]_{\mathcal{M}} \in \Sigma(\sim)$, hence $\theta(a)(m)([\![\psi]\!]_{\mathcal{M}}) = \theta(a)(m')([\![\psi]\!]_{\mathcal{M}})$.

The case $\phi = \phi_1|\phi_2$**:** $\mathcal{M}, m \Vdash \phi_1|\phi_2$ iff $m \equiv m_1\otimes m_2$ and $\mathcal{M}, m_i \Vdash \phi_i$, $i = 1,2$. From the transitivity of $\equiv$ we obtain $n \equiv m_1\otimes m_2$ and further $\mathcal{M}, n \Vdash \phi_1|\phi_2$.

The case $\phi = \phi_1 - \phi_2$**:** $\mathcal{M}, m \Vdash \phi_1 - \phi_2$ iff $\mathcal{M}, m_2 \Vdash \phi_2$ implies $\mathcal{M}, m\otimes m_2 \Vdash \phi_1$. But $m\otimes m_2 \equiv n\otimes m_2$ which, in the context described before, implies $\mathcal{M}, n \Vdash \phi_1 - \phi_2$.

Proof (Theorem 3). This proof reproduces at stochastic level the proof of Lemma 7.16 presented in [29] for probabilistic systems. Before starting the proof, we introduce some additional concepts and present some results that are needed for our proof. One of them is the *zigzag morphism*, similar to [12, 29], which is a functional analogue of the concept of bisimulation and will be the cornerstone of the completeness proof.
[Zigzag morphism] Given $\mathcal{M} = (M,\Sigma,\theta)$, $\mathcal{M}' = (M',\Sigma',\theta') \in \mathfrak{M}$, a *zigzag morphism* is a function $f : M \to M'$ that is surjective, measurable and for all $a \in \mathcal{A}$, $m \in M$ and $S' \in \Sigma'$,

$$\theta(a)(m)(f^{-1}(S')) = \theta'(a)(f(m))(S').$$

Because $\approx$ is an equivalence relation, we can take for a given (M,Σ) the quotient $(M^{\approx},\Sigma^{\approx})$ constructed as follows. $M^{\approx}$ is the set of all equivalence classes of M; there exists a projection $\pi : M \to M^{\approx}$ which maps each element to its equivalence class. π determines a σ-algebra $\Sigma^{\approx}$ on $M^{\approx}$ by $S \in \Sigma^{\approx}$ iff $\pi^{-1}(S) \in \Sigma$. We call π the *canonical projection* from (M,Σ) into $(M^{\approx},\Sigma^{\approx})$.

We state now a few results that allow us to prove the theorem.

Given a set X, a family of subsets $\Pi \subset 2^X$ closed under finite intersection is called π-*system*. A family of subsets $\Lambda \subset 2^X$ is a λ-*system* if contains X and is closed under complementation and countable union of pairwise disjoint sets.

[Dynkin's $\lambda - \pi$ theorem]: *If Π is a π-system and Λ is a λ-system, then $\Pi \subset \Lambda$ implies $\overline{\Pi} \subseteq \Lambda$, where $\overline{\Pi}$ is the σ-algebra generated by Π.*

Dynkin's theorem allows us to prove the next lemma.

[Lemma A.] *Suppose that $\Pi \subseteq 2^X$ is a π-system with $X \in \Pi$ and μ, ν are two measures on $(X, \overline{\Pi})$. If μ and ν agree on all the sets in Π, then they agree on $\overline{\Pi}$.*

We also present two more lemmas (see, e.g., [29] Section 7.7).

[Lemma B.] *Let (M, Σ) be an analytic set and let Σ_0 be a countably generated sub-σ-algebra of Σ which separates points in M, i.e., for any $m, n \in M$, $m \neq n$, there exists $S \in \Sigma_0$ such that $m \in S \not\ni n$. Then $\Sigma_0 = \Sigma$.*

[Lemma C.] *Let (M, Σ) be an analytic set and let $\equiv$ be an equivalence relation on M. If there exists a sequence $f_1, f_2, ...$ of real-valued Borel functions on M such that $m \equiv n$ iff for all i, $f_i(m) = f_i(n)$, then $(M^{\equiv}, \Sigma^{\equiv})$ is an analytic set.*

[Proposition D].*For any $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, there exists $\mathcal{M}^{\approx} = (M^{\approx}, \Sigma^{\approx}, \theta^{\approx}) \in \mathfrak{M}$ such that the canonical projection $\pi : (M, \Sigma, \theta) \to (M^{\approx}, \Sigma^{\approx}, \theta^{\approx})$ is a zigzag morphism.*

Now we prove Proposition D. For the beginning we show that $(M^{\approx}, \Sigma^{\approx})$ is an analytic set. Let $\mathcal{L}^*(\mathcal{A}) = \{\phi_i | i \in \mathbb{N}\}$. Because $[\![\phi_i]\!]_{\mathcal{M}}$ is measurable, the characteristic functions $1_{\phi_i} : M \to \{0, 1\}$ are measurable and $m \approx n$ iff $[\forall i \in \mathbb{N}, 1_{\phi_i}(m) = 1_{\phi_i}(n)]$. Lemma C proves further that $(M^{\approx}, \Sigma^{\approx})$ is an analytic set.

Let $\mathcal{B} = \{\pi([\![\phi_i]\!]_{\mathcal{M}} | i \in \mathbb{N}\}$. We show that $\overline{\mathcal{B}} = \Sigma^{\approx}$. Obviously, $\mathcal{B} \subseteq \Sigma^{\approx}$, because for any $\pi([\![\phi_i]\!]_{\mathcal{M}}) \in \mathcal{B}$, $\pi^{-1}(\pi([\![\phi_i]\!]_{\mathcal{M}})) \in \Sigma$. Notice that $\overline{\mathcal{B}}$ separates points in $M^{\approx}$: let $C, D \in M^{\approx}$, $C \neq D$ and let $m \in \pi^{-1}(C)$, $n \in \pi^{-1}(D)$; because $m \not\approx n$, there exists $\phi \in \mathcal{L}^*(\mathcal{A})$ such that $m \in [\![\phi]\!]_{\mathcal{M}} \not\ni n$. Hence, we can apply Lemma B and we obtain $\overline{\mathcal{B}} = \Sigma^{\approx}$.

Now we define $\theta^{\approx}$ such that π is a zigzag. Notice first that π is measurable and surjective by definition. For each $C \in \Sigma^{\approx}$ and $\alpha \in \mathcal{A}$, let $\theta^{\approx}(\alpha)(m^{\approx})(C) = \theta(\alpha)(m)(\pi^{-1}(C))$.

This definition is correct: let $m, n \in m^{\approx}$, we prove that $\theta(\alpha)(m)$ and $\theta(\alpha)(n)$ agree on $\Sigma^{\approx}$. We show first that they agree on $[\![\phi]\!]_{\mathcal{M}} \in \mathcal{B}$. Suppose that we have $\theta(\alpha)(m)([\![\phi]\!]_{\mathcal{M}}) < r < \theta(\alpha)([\![\phi]\!]_{\mathcal{M}})$. Then, $\mathcal{M}, m \Vdash \neg L_r^{\alpha}\phi$ while $\mathcal{M}, n \Vdash L_r^{\alpha}\phi$ - impossible. Because $\mathcal{B}$ is closed under finite intersection ($[\![\phi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} = [\![\phi \wedge \psi]\!]_{\mathcal{M}}$) and $M = [\![\top]\!]_{\mathcal{M}} \in \mathcal{B}$, we apply Lemma A and obtain that $\theta(\alpha)(m)$ and $\theta(\alpha)(n)$ agree on $\Sigma^{\approx}$.

Now we only need to prove that for any $\alpha \in \mathcal{A}$, $\theta^{\approx}(\alpha)$ is measurable. Let $C \in \Sigma^{\approx}$ and A a Borel set of $\mathbb{R}^+$. We have

$$(\theta^{\approx})^{-1}(\{\mu \in \Delta(M^{\approx}, \Sigma^{\approx}) | \mu(B) \in A\}) = \pi((\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\})).$$

But $\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\}$ is measurable in $\Delta(M, \Sigma)$ and because $\theta(\alpha)$ is measurable we obtain that $(\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\}) \in \Sigma$ implying $\pi((\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\})) \in \Sigma^{\approx}$. And this concludes the proof of Proposition D.

Now we have the ingredients to prove Theorem 3.

We prove that $\approx$ is a rate bisimulation. As before, it is sufficient to prove it for the case $\mathcal{M} = \mathcal{M}'$. Let $C \in \Sigma(\approx)$. Then, $C = \pi^{-1}(\pi(C))$, where π is the canonical projection. Because π is measurable, we get that $\pi(C) \in \Sigma^{\approx}$.

If $m \approx m'$, then $\pi(m) = \pi(m')$. Hence, $\theta(a)(m)(C) = \theta(a)(m)(\pi^{-1}(\pi(C))) = \theta^{\approx}(a)(\pi(m))(\pi(C))$, because π is a zigzag morphism. But $\theta^{\approx}(a)(\pi(m))(\pi(C)) = \theta^{\approx}(a)(\pi(m'))(\pi(C))$ and $\theta^{\approx}(a)(\pi(m'))(\pi(C)) = \theta(a)(m')(C)$. This proves that $\theta(a)(m)(C) = \theta(a)(m')(C)$ and it concludes the proof.

Proof (Lemma 3). We have that $\vdash \bigwedge_{f \in \mathcal{F}} (f \to 1)$ and using rule (R8) we obtain $\vdash \boxed{+}_{f_i \in \mathcal{F}}^{s \leq \bar{k}} f_1|...|f_s$. A consequence of applying (R8) is also that for any $\Pi_1, \Pi_2 \in \Omega[\psi]$, there exists a unique $\Pi \in \Omega[\psi]$ such that $\Pi_1|\Pi_2 \subseteq \Pi$.

For each consistent $\phi \in \mathcal{L}[\psi]$ there exists $k \leq \bar{k}$ such that $\vdash \phi \to k$. From the construction of the maximally-consistent sets we have $\vdash 1 \to \bigvee_{f \in \mathcal{F}} f$; we can apply (R7) and obtain that for each $k \in \mathbb{N}$, $\vdash k \to \bigvee_{f_i \in \mathcal{F}}^{s<k} f_1|...|f_s$. Hence, for each consistent $\phi \in \mathcal{L}[\psi]$ there exists $k \leq \bar{k}$ such that $\vdash \phi \to \bigvee_{f_i \in \mathcal{F}}^{s<k} f_1|...|f_s$. And because $\vdash \boxed{+}_{f_i \in \mathcal{F}}^{s \leq \bar{k}} f_1|...|f_s$, simple Boolean reasoning proves that there exists $f_1|...|f_s \in \mathcal{F}^*$ with $s \leq k$ such that $\vdash f_1|..|f_s \to \phi$. Hence, ϕ is contained in at least one set $\Pi \in \Omega[\psi]$ and $\mathcal{F}^*$ is the set of generators for the elements of $\Omega[\psi]$.

Proof (Lemma 4). (4) The existence of x and y derives from the construction of $\Omega_p[\psi]$ and the Rules (R2), (R3).
Because Γ is consistent and $L_x^a\phi, \neg L_y^a\phi \in \Gamma$, $x \neq y$. If $x > y$, $L_x^a\phi \in \Gamma$ entails (Axiom (A2)) $L_y^a\phi \in \Gamma$, contradicting the consistency of Γ.
Hence, $x < y$. If $x + 1/p < y$, then $L_{x+1/p}^a\phi \notin \Gamma$ (because $x < x + 1/p \in \mathbb{Q}_q$ and Γ is $\mathcal{L}_p$-maximally consistent), i.e. $\neg L_{x+1/p}^a\phi \in \Gamma$ implying that $x + 1/p \geq y$ - contradiction.

Proof (Lemma 5). As before, the existence of sup and inf is guaranteed by the construction and the Rules (R2) and (R3). Let $x^\infty = sup\{r \in \mathbb{Q} : L_r^a\phi \in \Gamma^\infty\}$ and $y^\infty = inf\{r \in \mathbb{Q} : \neg L_r^a\phi \in \Gamma^\infty\}$.

Suppose that $x^\infty < y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty < r < y^\infty$. This implies that $\neg L_r^a\phi \in \Gamma^\infty$ (from the definition of x^∞) and $L_r^a\phi \in \Gamma^\infty$ (from the definition of y^∞) - impossible because Γ^∞ is consistent.

Suppose that $x^\infty > y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty > r > y^\infty$. As Γ^∞ is maximally consistent we have either $L_r^a\phi \in \Gamma^\infty$ or $\neg L_r^a\phi \in \Gamma^\infty$. The first case contradicts the definition of x^∞ while the second the definition of y^∞.

Obviously, $x \leq x^\infty \leq y$. We cannot have $x^\infty = y$ because else $L_{x^\infty}^a\phi, \neg L_{x^\infty}^a\phi \in \Gamma$ contradicting the consistency of Γ.

Proof (Lemma 6). For the beginning we prove that $\mathcal{K}_\psi \in \mathfrak{K}$. This result is a direct consequence of the construction of $\mathcal{K}_\psi$. Firstly notice that because the space is discrete, is Polish, hence analytic set.

The central problem is to prove that for arbitrary $\Gamma \in \Omega_p[\psi]$ and $a \in \mathcal{A}$, the function $\theta_\psi(a)(\Gamma) : 2^{\Omega_p[\psi]} \to \mathbb{R}^+$ is well defined and a measure on $(\Omega_p[\psi], 2^{\Omega_p[\psi]})$. Next, we show that $\theta_\psi(a) \in [\![\Omega_p[\psi] \to \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]\!]$ and conclude the proof.

$\theta_\psi(a)(\Gamma)$ is well defined: suppose that for $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ we have $[\![\phi_1]\!] = [\![\phi_2]\!]$. Then, from Lemma 4, $\vdash \phi_1 \leftrightarrow \phi_2$ and from Rule (R1) $\vdash L_r^a\phi_1 \leftrightarrow L_r^a\phi_2$. This entails $a^\Gamma_{\phi_1} = a^\Gamma_{\phi_2}$ and guarantees that $\theta_\psi(a)(\Gamma)$ is well defined.

Now we prove that $\theta_\psi(a)(\Gamma)$ is a measure.

For showing $\theta_\psi(a)(\Gamma)(\emptyset) = 0$, we show that for any $r > 0$, $\vdash \neg L_r^a\bot$. This is sufficient, as Axiom (A1) guarantees that $\vdash L_0^a\bot$ and $[\![\bot]\!] = \emptyset$. Suppose that there exists $r > 0$ such that $L_r^a\bot$ is consistent. Let $\epsilon \in (0, r) \cap \mathbb{Q}$. Then Axiom (A2) gives $\vdash L_r^a\bot \to L_\epsilon^a\bot$. Hence, $\vdash L_r^a\bot \to (L_r^a(\bot \wedge \bot) \wedge L_\epsilon^a(\bot \wedge \neg\bot))$ and applying the Axiom (A3), $\vdash L_r^a\bot \to L_{r+\epsilon}^a\bot$. Repeating this argument we can prove that $\vdash L_r^a\bot \to L_s^a\bot$ for any s and Rule (R3) confirms the inconsistency of $L_r^a\bot$.

We show now that if $A, B \in 2^{\Omega_p[\psi]}$ with $A \cap B = \emptyset$, then $\theta_\psi(a)(\Gamma)(A) + \theta_\psi(a)(\Gamma)(B) = \theta_\psi(a)(\Gamma)(A \cup B)$. Let $A = [\![\phi_1]\!]$, $B = [\![\phi_2]\!]$ with $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ and $\vdash \phi_1 \to \neg\phi_2$. Let $x_1 = \theta_\psi(a)(\Gamma)(A)$, $x_2 = \theta_\psi(a)(\Gamma)(B)$ and $x = \theta_\psi(a)(\Gamma)(A \cup B)$. We prove that $x_1 + x_2 = x$.

Suppose that $x_1 + x_2 < x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x_1' + x_2' < x$, where $x_i' = x_i + \epsilon_i$ for $i = 1, 2$. But this implies that $L^a_{x_i'}\phi_i \notin \Gamma^\infty$ (from the definition of x_i), hence $\neg L^a_{x_i'}\phi_i \in \Gamma^\infty$. Further, using Axiom (A4), we obtain $\neg L^a_{x_1'+x_2'}(\phi_1 \vee \phi_2) \in \Gamma^\infty$, implying (from the definition of x) that $x_1' + x_2' \geq x$ - contradiction.

Suppose that $x_1 + x_2 > x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x_1'' + x_2'' > x$, where $x_i'' = x_i - \epsilon_i$ for $i = 1, 2$. But this implies (from the definition of x_i) that $L^a_{x_i''}\phi_i \in \Gamma^\infty$. Further, Axiom (A3) gives $L^a_{x_1''+x_2''}(\phi_1 \vee \phi_2) \in \Gamma^\infty$, i.e. $x_1'' + x_2'' \leq x$ - contradiction.

Now we prove that $\mathcal{M}_\psi \in \mathfrak{M}$.

From Remark 3 we know that $\otimes$ is welldefined. Associativity and commutativity is guaranteed by the axioms (A5) and (A6) while the structural granularity by Rule (R6) applied to $\mathcal{F}$.

It remains to prove that $(\mathcal{M}_\psi, \Gamma'\otimes\Gamma'') \sim (\mathcal{M}_\psi \times \mathcal{M}_\psi, (\Gamma', \Gamma''))$. This requires to prove that $\Gamma'|\Gamma'' \subseteq \Gamma$ implies that for arbitrary $\phi \in \mathcal{L}[\psi]$,

$$\theta_\psi(a)(\Gamma)([\![\phi]\!]) = \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g''\in\mathcal{F}^*} \theta_\psi(b)(\Gamma')([\![g']\!]) \bullet \theta_\psi(c)(\Gamma'')([\![g'']\!]).$$

Let $x = \theta_\psi(a)(\Gamma)([\![\phi]\!])$ and $x^b_{g'} = \theta_\psi(b)(\Gamma')([\![g']\!])$, $x^c_{g''} = \theta_\psi(c)(\Gamma'')([\![g'']\!])$. We need to prove that $x = \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g''\in\mathcal{F}^*} x^b_{g'} \bullet x^c_{g''}$.

We have $L^b_{x^b_{g'}} g' \in \Gamma'$ and $L^c_{x^c_{g''}} g'' \in \Gamma''$ for all $g', g'' \in \mathcal{F}^*$ such that $[\![\phi]\!] = [\![g'|g'']\!]$. But $[\![\phi]\!] = [\![g'|g'']\!]$ implies $\vdash \bigwedge_{[\![\phi]\!]=[\![g'|g'']\!]} g'|g'' \to \phi$ and because the elements of $\mathcal{F}^*$ are mutually disjoint and $\Gamma'|\Gamma'' \subseteq \Gamma$, we can apply Rule (R7) and obtain

$L^a_{(\sum_{b*c=a}\sum^{g',g''\in\mathcal{F}^*}_{[\![\phi]\!]=[\![g'|g'']\!]} x^b_{g'}\bullet x^c_{g''})}\phi \in \Gamma$. Hence,

$$x \geq \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g''\in\mathcal{F}^*} x^b_{g'} \bullet x^c_{g''}.$$

Suppose that $x > \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g''\in\mathcal{F}^*} x^b_{g'} \bullet x^c_{g''}$. Because $\bullet$ is continuous, there exist $\epsilon^b_{g'}, \epsilon^c_{g''} > 0$ such that

$$x > \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g''\in\mathcal{F}^*} (x^b_{g'} + \epsilon^b_{g'}) \bullet (x^c_{g''} + \epsilon^c_{g''}).$$

We have $\neg L^b_{x^b_{g'}+\epsilon^b_{g'}} g' \in \Gamma'$ and $\neg L^c_{x^c_{g''}+\epsilon^c_{g''}} g'' \in \Gamma''$ for all $g', g'' \in \mathcal{F}^*$ such that $[\![\phi]\!] = [\![g'|g'']\!]$. But $[\![\phi]\!] = [\![g'|g'']\!]$ also implies $\vdash \bigwedge_{[\![\phi]\!]=[\![g'|g'']\!]} \phi \to g'|g''$ and because the elements of $\mathcal{F}^*$ are mutually disjoint and $\Gamma'|\Gamma'' \subseteq \Gamma$, we can apply Rule (R8) and obtain $\neg L^a_{(\sum_{b*c=a}\sum^{g',g''\in\mathcal{F}^*}_{[\![\phi]\!]=[\![g'|g'']\!]} x^b_{g'}\bullet x^c_{g''})}\phi \in \Gamma$. Hence,

$$x < \sum_{b*c=a} \sum_{[\![\phi]\!]=[\![g'|g'']\!]}^{g',g''\in\mathcal{F}^*} (x^b_{g'} + \epsilon^b_{g'}) \bullet (x^c_{g''} + \epsilon^c_{g''}), \text{ - contradiction!}$$