Continuous Markovian Logic

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Abstract. In this paper we introduce Continuous Markovian Logic (CML), a simple formalism inspired by coalgebraic logic, to characterize the bisimulation of Markov processes having continuous state space and continuous temporal evolution (CMPs). The alternative, continuous stochastic logic (CSL), has expressive power which comes at the price of a complicated two-layer semantics and requires a complex mathematical apparatus for calculating the probabilities of computational paths in time. CML focuses on the rates of the exponentially distributed random variables that characterize the duration of transitions, instead of explicitly expressing time-dependent probabilistic information as in the case of CSL. Thus, CML exploits the fact that the probabilistic-temporal information is encapsulated in the rates. We prove that the negation-free fragment of CML is already sufficient to completely characterize bisimulation of CMPs. We also show a sound-complete axiomatization of CML and demonstrate that, unlike CSL, it enjoys the small model property.

1 Introduction

Many complex natural and man-made systems (e.g., biological, ecological, physical, social, financial, and computational) are modeled as stochastic processes. The stochasticity is used to represent both a lack of knowledge and inherent randomness. These systems are frequently studied both in interaction with discrete systems, such as controllers, or with interactive environments having continuous behavior. This context has motivated research aiming to develop a general theory of systems that is able to uniformly treat discrete, continuous and hybrid interactive systems. Two of the central questions of this research are *"when do two systems behave the same (are bisimilar)?"* and *"is there any (algorithmic) technique to check the bisimuation of two systems?"*. These both questions are related to the problems of state reduction (collapsing a model to an equivalent reduced model) and discretization (reduce a continuous or hybrid system to an equivalent discrete one) which are cornerstones in the field of stochastic systems.

In this paper we prove that, for the most general case of continuous time and continuous space Markovian processes (CMPs), two systems are bisimilar iff they satisfy the same formulas of a particular logic that we define: continuous Markovian logic (CML). Thus, the satisfiability relation of CML provides a test for bisimulation and addresses the above mentioned questions. Previously, it has been proved that continuous stochastic logic (CSL) [2] characterizes the bisimulation of CMPs [3, 11], but the price of its expressive power is a complex semantics involving both state and path formulas and a complicated mathematical apparatus used to calculate the probability of computational paths. CML, on the other hand, is a much simpler logic that explores the coalgebraic structure of Markovian processes, using operators that speak about rates[1] only. In effect, the probabilistic-temporal information expressed explicitly by CSL is *compressed* in CML. For instance, instead of expressing that *"with probability at least* $p \in [0,1]$*, within time* t *the system will reach a next state satisfying* ϕ*"* (which can be expressed in CSL by the formula $\mathcal{P}_{\geq p}X^{[0,t]}\phi$), one can get this probability from the rate $r \in \mathbb{R}_+$ of the transition: the cumulative distribution function $f(t) = 1 - e^{-rt}$ calculates the probability of the event ϕ within time t. The formula $L_r\phi$ of CML is interpreted as *"the rate of the event* ϕ *is at least* r*"*.

We prove that the negation-free fragment of CML is expressive enough to completely characterize the bisimulation of CMPs and is thus appropriate to do *"the job"* of CSL. CML is less expressive than CSL, being unable to express path properties such as, for instance, oscillations. This paper shows, however, that we do not need to check such properties in order to decide bisimulation. This is both surprising (because bisimulation refers also to infinite behaviours) and at the same time expected, since similar situations have been proved for nondetermnistic and probabilistic cases.

We also present a complete axiomatization for CML and prove that, unlike CSL, it enjoys the finite model property.

Research context. In a series of foundational papers dedicated to probabilistic systems [18, 10, 6, 9] Panangaden et al. studied *labeled Markov processes* (LMPs) – general mathematical structures encapsulating both discrete and continuous state space probabilistic models. LMPs work with probability densities: the state space is measurable and probability distributions associate to each tuple (*state, measurable set*) the probability of a transition from that state to some state of the set; the distributions are labeled in order to differentiate various effects of the interactions with the environment. In [4, 9] it was shown that the bisimulation of LMPs – an extension of the probabilistic bisimulation defined in [16] for discrete systems – can be logically characterized by the negation-free fragment of *Hennessy-Milner logic* (H-ML) with modal operators indexed by transition labels and probabilities.

Similar probabilistic structures, *Harsanyi type spaces* (HTSs) [13, 17], are used in game theory and economy to model belief systems in societies of agents. As for LMPs, a logic has been developed – Aumann's system [1, 14, 21, 12] – to express qualitative and quantitative properties of HTSs. Unlike H-ML, however, Aumann's system does not involve operators with (modal: existential/universal) quantifications on the possible transitions; instead, Aumann's operators characterize *en bloc* the set of possible transitions, having a similar semantics to the Nabla operator of coalgebraic logic.

[1] The rate of a transition is the parameter of an exponentially distributed random variable that characterizes, for Markovian processes, the duration of the transition.

Markovian systems with discrete states and continuous time evolution are commonly known as *continuous-time Markov chains* (CTMCs) and have been intensively studied. For specifying their properties, continuous stochastic logic (CSL) has been developed [2] together with model-checking techniques [15]. The bisimulation of CTMCs, generalizing the bisimulation of discrete LMPs, can be characterized using CSL [3] and this characterization is complete [11].

In the paper proving that CSL characterizes the bisimulation of CTMCs [11], the authors actually prove a more general result: CSL characterizes the bisimulation of CMPs.

Our approach. Based on an equivalence between the definitions of HTSs and LMPs evidenced by Doberkat [7], we extend Aumann's system for CMPs and obtain CML. Instead of the path formulas and the operators indexed with time intervals and probabilities of CSL, CML has only state formulas and operators indexed with the rates of transitions. From this point of view, CML is similar to H-ML used in [4, 9] and actually the proof of the complete characterization of bisimulation follows the general method used in [18] with H-ML. On the other hand, unlike H-ML, CML uses no quantification on possible transitions. The similarity of CML to Aumann's system has allowed us to identify a Hilbert-style complete axiomatization for CML related to the one presented in [21] for HTSs.

The structure of the paper. The first section comprises some preliminary definitions. Section 3 introduces CMPs and their bisimulation. In Section 4 we define the logic CML. Section 5 studies the relation between the logical equivalence induced by CML and the bisimulation of CMPs, concluding that the negation-free fragment of CML completely characterizes bisimulation. In Section 6 we present a sound-complete axiomatization for CML proving, at the same time, the small model property. We also have a conclusive section and an Appendix with additional definitions and proofs.

2 Preliminary definitions

In this section we introduce some notations and recall a few notions of measure theory to establish the terminology used in the paper.

For arbitrary sets A and B, 2^A denotes the powerset of A, $A \uplus B$ their disjoint union and B^A the class of functions from A to B. If $f \in B^A$ we denote by $f^{-1} : 2^B \to 2^A$ the inverse mapping of f.

For non null $n \in \mathbb{N}$, let $\mathbb{Q}_n = \{\frac{p}{n} : p \in \mathbb{N}\}$. If $S \subseteq \mathbb{Q}$ is finite, then *granularity of* S, $gran(S)$, is the lowest common denominator of the elements of S.

Given a set M, a *σ-algebra Σ* over M is a subsets of 2^M containing M and closed under complement and countable union. The tuple (M, Σ) is called a *measurable space*, the elements of Σ *measurable sets* and M the *support-set*.

A set $\Omega \in 2^M$ is a *generator for the σ-algebra Σ* on M if Σ is the closure of Ω under complement and countable union; we write $\overline{\Omega} = \Sigma$ and say that Σ is generated by Ω. A generator Ω for Σ is *a base of* Σ if it has disjoint elements.

A *measure* on a measurable space $\mathcal{M} = (M, \Sigma)$ is a function $\mu : \Sigma \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$ and for any $\{N_i | i \in I \subseteq \mathbb{N}\} \subseteq \Sigma$ with pairwise disjoint elements, $\mu(\bigcup_{i\in I} N_i) = \sum_{i\in I} \mu(N_i)$.

If Ω is a base for (M, Σ), $N \in \Omega$ and $r \in \mathbb{R}^+$, then $D(r, N)$ is the *r-Dirac measure on* N, which is the measure induced by the function

$$f(N') = \begin{cases} r \text{ if } N' = N \\ 0 \text{ if } N' \neq N \end{cases}$$

Let $\Delta(M, \Sigma)$ be the class of measures on (M, Σ). We organize it as a measurable space by considering the σ-algebra generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets $\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$.

Given two measurable spaces (M, Σ) and (N, Θ), a mapping $f : M \to N$ is *measurable* if for any $T \in \Theta, f^{-1}(T) \in \Sigma$. We use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Θ).

The notion of an *analytic space* is central to the definition of continuous Markov processes and is in fact a necessity for technical reasons[2]. This requirement does not influence the general presentation and hence we only sketch the main definitions. For a detailed discussion the reader is referred to [18] (Section 7.5) or to [7] (Section 4.4).

A metric space (M, d) is *complete* if every Cauchy sequence converges in M.

A *Polish space* is the topological space underlying a complete metric space with a countable dense subset. Note that any discrete space is Polish.

An *analytic space* is the image of a Polish space under a continuous function from one Polish space to another. Note that any Polish space is analytic.

3 Continuous Markov processes

In this section we introduce $\mathcal{A}$-continuous Markov processes ($\mathcal{A}$-CMPs) as coalgebraic structures [19] that encode stochastic behaviors. The labels $\alpha \in \mathcal{A}$ represent types of interactions with the environment. If m is the current state of the system and N is a measurable set of states, the function $\theta(\alpha)(m)$ is a measure on the state space and $\theta(\alpha)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variabe that characterizes the duration of an α-transition from m to arbitrary $n \in N$. Indeterminacy in such systems is resolved by races between events executing at different rates.

Definition 1 (Continuous Markov process). *Given an analitic space* (M, Σ), *where* Σ *is the Borel algebra generated by the topology, and a denumerable set* $\mathcal{A}$, *an* $\mathcal{A}$-continuous Markov process *is a tuple* $\mathcal{M} = (M, \Sigma, \theta)$, *where*

$$\theta : \mathcal{A} \to [\![M \to \Delta(M, \Sigma)]\!].$$

[2] Some of the proofs are using the fact that the quotients of analytic spaces are themselves analytic.

Notice that $\theta(\alpha)$ is defined as a measurable mapping between (M, Σ) and the measurable space of the measures on (M, Σ). This condition is equivalent to the conditions on the two-variable *rate function* used in [11] to define transitions for CMPs (see, e.g. Proposition 2.9, of [7]).

We define the stochastic bisimulation relation on CMPs following the similar definition given in [9] for LMPs.

Given a binary relation $\mathfrak{R} \subseteq M \times M$ on a set M, we call a subset $N \subseteq M$ $\mathfrak{R}$-closed iff $\{m \in M \mid \exists n \in N, (n,m) \in \mathfrak{R}\} \subseteq N$. If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

Definition 2 (Bisimulations on CMPs). *Given an $\mathcal{A}$-CMP $\mathcal{M} = (M, \Sigma, \theta)$, a* rate-bisimulation relation *on it is an equivalence relation $\mathfrak{R} \subseteq M \times M$ such that $(m,n) \in \mathfrak{R}$ iff for any $C \in \Sigma(\mathfrak{R})$ and any $\alpha \in \mathcal{A}$,*

$$\theta(\alpha)(m)(C) = \theta(\alpha)(n)(C).$$

Two elements $m, n \in M$ are stochastic bisimilar*, written $m \sim_{\mathcal{M}} n$, if they are related by a rate-bisimulation relation.*

Observe that, for any $\mathcal{A}$-CMP (M, Σ, θ), there exist rate-bisimulation relations. For instance, the identity of the elements of M is a rate-bisimulation relation.

In what follows we extend the definition of bisimulation from states of the same system to different systems. As in [9, 11, 18] we do this by considering the disjoint union of systems. It is trivial to check that the definition is correct.

Given two $\mathcal{A}$-CMPs (M, Σ, θ) and (M', Σ', θ'), their *disjoint union* is the $\mathcal{A}$-CMP $(M'', \Sigma'', \theta'')$ denoted $(M, \Sigma, \theta) \uplus (M', \Sigma', \theta')$ and defined by $M'' = M \uplus M'$, $\Sigma'' = \overline{\Sigma \uplus \Sigma'}$ and for $\alpha \in \mathcal{A}$, $N \in \Sigma$ and $N' \in \Sigma'$,

$$\theta''(\alpha)(m)(N \uplus N') = \begin{cases} \theta(\alpha)(m)(N) & \text{if } m \in M \\ \theta'(\alpha)(m)(N') & \text{if } m \in M' \end{cases}$$

Given two $\mathcal{A}$-CMPs $\mathcal{M} = (M, \Sigma, \theta)$ and $\mathcal{M}' = (M', \Sigma', \theta')$, $m \in M$ and $m' \in M'$, we say that $(m, \mathcal{M})$ and $(m', \mathcal{M}')$ are *bisimilar* written $(m, \mathcal{M}) \sim (m', \mathcal{M}')$ iff $m \sim_{\mathcal{M} \uplus \mathcal{M}'} m'$.

4 Continuous Markovian Logic

In this section we introduce continuous Markovian logic (CML) for semantics based on $\mathcal{A}$-CMPs. This logic extends Aumann's system [1] to stochastic domains.

Given a denumerable set $\mathcal{A}$, the formulas of CML are the elements of the set $\mathcal{L}(\mathcal{A})$ introduced by the following grammar, for arbitrary $\alpha \in \mathcal{A}$ and $r \in \mathbb{Q}_+$.

$$\phi := \top \mid \neg\phi \mid \phi \wedge \phi \mid L_r^{\alpha}\phi.$$

We denote by $\mathcal{L}^*(\mathcal{A})$ the negation-free fragment of $\mathcal{L}(\mathcal{A})$.

The semantics of CML is given by the *satisfiability relation* defined, for an arbitrary $\mathcal{A}$-CMP $\mathcal{M} = (M, \Sigma, \theta)$ and arbitrary $m \in M$, inductively as follows.

$\mathcal{M}, m \Vdash \top$ always,

$\mathcal{M}, m \Vdash \neg\phi$ iff it is not the case that $\mathcal{M}, m \Vdash \phi$,

$\mathcal{M}, m \Vdash \phi \wedge \psi$ iff $\mathcal{M}, m \Vdash \phi$ and $\mathcal{M}, m \Vdash \psi$,

$\mathcal{M}, m \Vdash L_r^\alpha \phi$ iff $\theta(\alpha)(m)([\![\phi]\!]_\mathcal{M}) \geq r$,

where $[\![\phi]\!]_\mathcal{M} = \{m \in M | \mathcal{M}, m \Vdash \phi\}$.

The formula $L_r^\alpha \phi$ is interpreted as *"the rate of an α transition from the current state to a state satisfying ϕ is at least r"*. Notice that the semantics of $L_r^\alpha \phi$ is well defined only if $[\![\phi]\!]_\mathcal{M}$ is measurable. This is guaranteed by the fact that $\theta(\alpha)$ is a measurable mapping.

Lemma 1. *For any $\phi \in \mathcal{L}(\mathcal{A})$ and any $\mathcal{A}$-CMP $\mathcal{M} = (M, \Sigma, \theta)$, $[\![\phi]\!]_\mathcal{M} \in \Sigma$.*

Proof. Induction on the structure of ϕ. The only nontrivial case is $\phi = L_r^\alpha \psi$. Observe that $[\![L_r^\alpha \psi]\!]_\mathcal{M} = (\theta(\alpha))^{-1}(\{\mu \in \Delta(M, \Sigma) | \mu([\![\psi]\!]_\mathcal{M}) \geq r\})$. From the inductive hypothesis, $[\![\psi]\!]_\mathcal{M} \in \Sigma$, hence, $\{\mu \in \Delta(M, \Sigma) | \mu([\![\psi]\!]_\mathcal{M}) \geq r\}$ is measurable in $\Delta(M, \Sigma)$ and because $\theta(\alpha)$ is a measurable mapping, we obtain that $[\![L_r^\alpha \psi]\!]_\mathcal{M}$ is measurable.

When it is not the case that $\mathcal{M}, m \Vdash \phi$, we write $\mathcal{M}, m \nVdash \phi$. We also consider $\bot = \neg\top$. Notice that always $\mathcal{M}, m \nVdash \bot$.

A formula ϕ is *satisfiable* if there exist an $\mathcal{A}$-CMP $\mathcal{M} = (M, \Sigma, \theta)$ and $m \in M$ such that $\mathcal{M}, m \Vdash \phi$. ϕ is *valid in* $\mathcal{M}$, denoted $\mathcal{M} \Vdash \phi$, if for any $m \in M$, $\mathcal{M}, m \Vdash \phi$. ϕ is *valid*, denoted by $\Vdash \phi$, if $\neg\phi$ is not satisfiable.

5 Logical equivalence versus stochastic bisimulation

In this section we prove that the logical equivalence induced by CML characterizes stochastic bisimulation. Moreover, we show that the negation-free fragment already provides a complete characterization. The proof follows the general proof method used in [9, 11, 18] for similar results.

The next theorem states that bisimulation preserves the satisfiability of $\mathcal{L}(\mathcal{A})$ formulas.

Theorem 1 (Sound characterization of bisimulation). *Let $\mathcal{M} = (M, \Sigma, \tau)$ and $\mathcal{M}' = (M', \Sigma', \tau')$ be $\mathcal{A}$-CMPs and $m \in M$, $m' \in M'$. If $(\mathcal{M}, m) \sim (\mathcal{M}', m')$, then [for any $\phi \in \mathcal{L}(\mathcal{A})$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}', m' \Vdash \phi$].*

Proof. Induction on the structure of ϕ. We can assume, without loosing generality, that $\mathcal{M} = \mathcal{M}' = (M, \Sigma, \theta)$. The only nontrivial case is $\phi = L_r^\alpha \psi$.
$\mathcal{M}, m \Vdash \phi$ iff $\theta(\alpha)(m)([\![\psi]\!]_\mathcal{M}) \geq r$. From the inductive hypothesis we have $[\![\psi]\!]_\mathcal{M} \in \Sigma(\sim)$, hence $m \sim m'$ implies $\theta(\alpha)(m)([\![\psi]\!]_\mathcal{M}) = \theta(\alpha)(m')([\![\psi]\!]_\mathcal{M})$ implying $\theta(\alpha)(m')([\![\psi]\!]_\mathcal{M}) \geq r$, i.e., $\mathcal{M}, m' \Vdash \phi$.

Further we prove that the logical equivalence induced by the negation-free fragment $\mathcal{L}^*(\mathcal{A})$ of CML characterizes completely the stochastic bisimulation of CMPs. For simplicity, consider for arbitrary $\mathcal{A}$-CMPs $\mathcal{M} = (M, \Sigma, \tau)$, $\mathcal{M}' = (M', \Sigma', \tau')$ and $m \in M$, $m' \in M'$ the relation $\approx$ defined by $(\mathcal{M}, m) \approx (\mathcal{M}', m')$ iff [for any $\phi \in \mathcal{L}^*(\mathcal{A})$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}', m' \Vdash \phi$]. We will show that if $(\mathcal{M}, m) \approx (\mathcal{M}', m')$, then $(M, m) \sim (\mathcal{M}', m')$. For this, we show that $\approx$ is a rate-bisimulation.

For the beginning we introduce the concept of *zigzag morphism* for CMPs, similar to [9, 18], which is a functional analogue of the concept of bisimulation and will be the cornerstone of the completeness proof.

Definition 3 (Zigzag morphism). *A function f from $\mathcal{M} = (M, \Sigma, \theta)$ to $\mathcal{M}' = (M', \Sigma', \theta')$ is a zigzag morphism if it is surjective, measurable and for all $\alpha \in \mathcal{A}$, $m \in M$ and $S' \in \Sigma'$,*

$$\theta(\alpha)(m)(f^{-1}(S')) = \theta'(\alpha)(f(m))(S').$$

Notice that $\approx$ is an equivalence relation, hence, for a given (M, Σ) we can consider the quotient $(M^{\approx}, \Sigma^{\approx})$ constructed as follows. $M^{\approx}$ is the set of all equivalence classes of M; there exists a projection $\pi : M \to M^{\approx}$ which maps each element to its equivalence class. π determines a σ-algebra $\Sigma^{\approx}$ on $M^{\approx}$ by $S \in \Sigma^{\approx}$ iff $\pi^{-1}(S) \in \Sigma$. We call π the *canonical projection* from (M, Σ) into $(M^{\approx}, \Sigma^{\approx})$.

We state now the main lemma that allow us to prove the completeness theorem. The proof of this lemma goes similarly to the one of Lemma 7.16 presented in [18].

Lemma 2. *For any $\mathcal{A}$-CMP $\mathcal{M} = (M, \Sigma, \theta)$ there exists $\theta^{\approx} : \mathcal{A} \to [\![M^{\approx} \to \Delta(M^{\approx}, \Sigma^{\approx})]\!]$ such that $\mathcal{M}^{\approx} = (M^{\approx}, \Sigma^{\approx}, \theta^{\approx})$ is an $\mathcal{A}$-CMP and the canonical projection $\pi : (M, \Sigma, \theta) \to (M^{\approx}, \Sigma^{\approx}, \theta^{\approx})$ is a zigzag morphism.*

Now we are ready to prove the completeness theorem.

Theorem 2 (Complete characterization of bisimulation). *Let $\mathcal{M} = (M, \Sigma, \theta)$ and $\mathcal{M}' = (M', \Sigma', \theta')$ be two $\mathcal{A}$-CMPs and $m \in M$, $m' \in M'$. If $(\mathcal{M}, m) \approx (\mathcal{M}', m')$, then $(\mathcal{M}, m) \sim (\mathcal{M}', m')$.*

Proof. We prove that $\approx$ is a rate bisimulation. As before, it is sufficient to prove it for the case $\mathcal{M} = \mathcal{M}'$. Let $C \in \Sigma(\approx)$. Then, $C = \pi^{-1}(\pi(C))$, where π is the canonical projection. Because π is measurable, we get that $\pi(C) \in \Sigma^{\approx}$.

If $m \approx m'$, then $\pi(m) = \pi(m')$. Hence, $\theta(\alpha)(m)(C) = \theta(\alpha)(m)(\pi^{-1}(\pi(C))) = \theta^{\approx}(\alpha)(\pi(m))(\pi(C))$, because π is a zigzag morphism. But $\theta^{\approx}(\alpha)(\pi(m))(\pi(C)) = \theta^{\approx}(\alpha)(\pi(m'))(\pi(C))$ and $\theta^{\approx}(\alpha)(\pi(m'))(\pi(C)) = \theta(\alpha)(m')(C)$. This proves that $\theta(\alpha)(m)(C) = \theta(\alpha)(m')(C)$ and concludes the proof.

6 A complete Hilbert-style axiomatization for CML

Table 1 contains a Hilbert-style axiomatization for CML (to be considered in addition to the axiomatization of propositional logic). The axioms and rules are given for propositional variables $\phi, \psi \in \mathcal{L}(\mathcal{A})$ and for arbitrary $\alpha \in \mathcal{A}$ and $s, r \in \mathbb{Q}^+$. This system has some similarities to the one proposed in [21] for Harsanyi type spaces. As in the probabilistic case, we have two infinitary rules that encode the Archimedean properties of $\mathbb{Q} \cup \{+\infty\}$. However, we believe that a finitary axiomatization is possible, probably on the lines of [14].

(A1): $\vdash L_0^\alpha \phi$
(A2): $\vdash L_{r+s}^\alpha \phi \rightarrow L_r^\alpha \phi$
(A3): $\vdash L_r^\alpha(\phi \wedge \psi) \wedge L_s^\alpha(\phi \wedge \neg\psi) \rightarrow L_{r+s}^\alpha \phi$
(A4): $\vdash \neg L_r^\alpha(\phi \wedge \psi) \wedge \neg L_s^\alpha(\phi \wedge \neg\psi) \rightarrow \neg L_{r+s}^\alpha \phi$
(R1): If $\vdash \phi \rightarrow \psi$ then $\vdash L_r^\alpha \phi \rightarrow L_r^\alpha \psi$
(R2): If for all $r < s, \vdash \phi \rightarrow L_r^\alpha \psi$ then $\vdash \phi \rightarrow L_s^\alpha \psi$
(R3): If for all $r, \vdash \phi \rightarrow L_r^\alpha \psi$ then $\vdash \phi \rightarrow \bot$

Table 1. The axiomatic system of $\mathcal{L}(\mathcal{A})$

We say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the given axioms and rules. We say that ϕ is *consistent*, if $\phi \rightarrow \bot$ is not provable. Given a set Φ of formulas, we say that Φ proves ϕ if from the formulas of Φ and the axioms we can prove ϕ using the rules; we write $\Phi \vdash \phi$. Φ is consistent if it is not the case that $\Phi \vdash \bot$. If Φ is finite we denote by $\bigwedge \Phi = \bigwedge_{\phi \in \Phi} \phi$. For a sublanguage $\mathcal{L} \subseteq \mathcal{L}(\mathcal{A})$, we call Φ *$\mathcal{L}$-maximally consistent* if Φ is consistent and no formula of $\mathcal{L}$ can be added to it without making it inconsistent.

Theorem 3 (Soundness). *The axiomatic system of CML is sound for the semantics based on $\mathcal{A}$-CMPs, i.e., for any $\phi \in \mathcal{L}(\mathcal{A})$, if $\vdash \phi$ then $\Vdash \phi$.*

In what follows we prove that this axiomatic system is also complete with respect to the semantics introduced before, i.e., that all the validities are provable. For doing this it is sufficient to prove that any arbitrary consistent formula has a model. This is a general result that only depend on provability, consistency and satisfiability means.

Lemma 3. *If [for any ϕ, the consistency of ϕ implies that ϕ is satisfiable], then the logic is complete, i.e. for any ψ, $\Vdash \psi$ implies $\vdash \psi$.*

Before proceeding with the completeness proof we fix some notations.

Let the *modal depth of a formula* in $\mathcal{L}(\mathcal{A})$ be defined inductively as follows: $md(\top) = 0$, $md(\neg\phi) = md(\phi)$, $md(\phi_1 \wedge \phi_2) = max(md(\phi_1), md(\phi_2))$ and $md(L_r^\alpha \phi) = md(\phi) + 1$.

For arbitrary $A \subseteq \mathcal{A}$ and non null $n \in \mathbb{N}$, let $\mathcal{L}_n(A)$ be the sublanguage of $\mathcal{L}(\mathcal{A})$ that uses only modal operators L_r^α with $\alpha \in A$ and $r \in \mathbb{Q}_n$. If $\Lambda \subseteq \mathcal{L}(\mathcal{A})$, let $[\Lambda]_n = \{\phi \in \mathcal{L}_n(\mathcal{A}) : \Lambda \vdash \phi\}$.

We fix a formula $\psi \in \mathcal{L}(\mathcal{A})$. Under the hypothesis that ψ is consistent, we construct a model for it. Let $A \subset \mathcal{A}$ be the set of labels[3] of the modal operators of ψ. Let $n \in \mathbb{N}$ be the lowest common denominator of the indexes of the operators of ψ (*granularity of* ψ) and $m \in \mathbb{Q}_n$ the greatest index of the modal operators of ψ. Let $\mathcal{L}[\psi]$ be the sublanguage of $\mathcal{L}_n(A)$ that uses only modal operators with indexes no bigger than m and contains no formula with modal depth bigger than $md(\psi)$. All these notations are fixed for the rest of the section.

Let $\Omega[\psi]$ be the set of $\mathcal{L}[\psi]$-maximally consistent sets of formulas. By construction, $\Omega[\psi]$ and any $\Lambda \in \Omega[\psi]$ are finite sets.

For each $\Lambda \in \Omega[\psi]$, we construct an upper set of formulas $\Lambda^+ \supseteq [\Lambda]_n$ as follows. Suppose that $\Lambda = \{\phi_1, ..., \phi_k\}$.

Consider ϕ_1.

There exists $r \in \mathbb{Q}_n$ such that $[\Lambda]_n \cup \{\neg L_r^\alpha \phi_1\}$ is consistent (suppose that this is not the case, then $\vdash \bigwedge \Lambda \to L_r^\alpha \phi_1$ for all $r \in \mathbb{Q}_n$ hence, Rule (R3) proves that $\bigwedge \Lambda$ inconsistent - impossible).

Let $y_1^\alpha = min\{s \in \mathbb{Q}_n : [\Lambda]_n \cup \{\neg L_s^\alpha \phi_1\}$ is consistent$\}$ and $x_1^\alpha = max\{s \in \mathbb{Q}_n : L_s^\alpha \phi_1 \in [\Lambda]_n\}$ (this max exists because else, $\vdash \bigwedge \Lambda \to L_r^\alpha \phi_1$ for all r and Rule (R3) implies $\bigwedge \Lambda$ inconsistent - impossible).

Obviously, $x_1^\alpha < y_1^\alpha$. There exists $r \in \mathbb{Q}$, $x_1^\alpha < r < y_1^\alpha$ such that $\{\neg L_r^\alpha \phi_1\} \cup [\Lambda]_n$ is consistent (suppose that this is not the case, then $\vdash \bigwedge \Lambda \to L_r^\alpha \phi_1$ for all $r < y_1^\alpha$; hence, from Rule R2, $\vdash \bigwedge \Lambda \to L_{y_1^\alpha}^\alpha \phi_1$ - contradiction with the consistency of Λ). Obviously, $r \notin \mathbb{Q}_n$. Let $n_1 = gran\{1/n, r\}$.

If there exists $s \in \mathbb{Q}_{n_1}$, $\neg L_s^\alpha \phi_1 \in [\Lambda]_{n_1}$, take $\Lambda_1^\alpha = \Lambda$; else, let $s_1^\alpha = min\{s \in \mathbb{Q}_{n_1} : [\Lambda]_{n_1} \cup \{\neg L_s^\alpha \phi_1\}$ is consistent$\}$ and $\Lambda_1^\alpha = \Lambda \cup \{\neg L_{s_1}^\alpha \phi_1\}$.

Let $\Lambda_1 = \bigcup_{\alpha \in A} \Lambda_1^\alpha$. This set is consistent and is constructed in a finite number of steps because A is finite.

Consider ϕ_2.

As in the case of ϕ_1, let $y_2^\alpha = min\{s \in \mathbb{Q}_{n_1} : [\Lambda_1]_{n_1} \cup \{\neg L_s^\alpha \phi_2\}$ is consistent$\}$ and $x_2^\alpha = max\{s \in \mathbb{Q}_{n_1} : L_s^\alpha \phi_2 \in [\Lambda_1]_{n_1}\}$.

We have $x_2^\alpha < y_2^\alpha$ and there exists $r \in \mathbb{Q} \setminus \mathbb{Q}_{n_1}$, $x_2^\alpha < r < y_2^\alpha$ such that $\{\neg L_r^\alpha \phi_2\} \cup [\Lambda_1]_{n_1}$ is consistent. Let $n_2 = gran\{1/n_1, r\}$.

If there exists $s \in \mathbb{Q}_{n_2}$, $\neg L_s^\alpha \phi_2 \in [\Lambda_1]_{n_2}$, take $\Lambda_2^\alpha = \Lambda_1$; else, let $s_2^\alpha = min\{s \in \mathbb{Q}_{n_2} : [\Lambda_1]_{n_2} \cup \{\neg L_s^\alpha \phi_2\}$ is consistent$\}$ and $\Lambda_2^\alpha = \Lambda_1 \cup \{\neg L_{s_2}^\alpha \phi_2\}$.

Let $\Lambda_2 = \bigcup_{\alpha \in A} \Lambda_2^\alpha$.

We continue this construction for all k formulas in Λ and we obtain

$$\Lambda \subseteq \Lambda_1 \subseteq ... \subseteq \Lambda_k,$$

where Λ_k is a finite and consistent set of formulas. Let $n_\Lambda = gran\{1/n_1, .., 1/n_k\}$. We make this construction for any $\Lambda \in \Omega[\psi]$. Let $p = gran\{1/n_\Lambda : \Lambda \in \Omega[\psi]\}$. Notice that $p > n$. We denote by $\Lambda^+ = [\Lambda_k]_p$ for any $\Lambda \in \Omega[\psi]$ and by $\Omega^+[\psi] = \{\Lambda^+ : \Lambda \in \Omega[\psi]\}$.

Remark 1. Any consistent formula $\phi \in \mathcal{L}[\psi]$ is an element of a set $\Lambda^+ \in \Omega^+[\psi]$. Moreover, for any Λ^+ there exists a formula ρ such that $\phi \in \Lambda^+$ iff $\vdash \rho \to \phi$; for instance, $\rho = \bigwedge \Lambda_k$.

[3] Observe that A is finite.

Let Ω_p be the set of $\mathcal{L}_p(A)$-maximally consistent sets of formulas. We fix a projection[4] $\sigma : \Omega^+[\psi] \to \Omega_p$, i.e., an injection such that for any $\Lambda^+ \in \Omega^+[\psi]$, $\Lambda^+ \subseteq \sigma(\Lambda^+)$. We denote by $\Omega_p[\psi] = \sigma(\Omega^+[\psi])$. For $\phi \in \mathcal{L}[\psi]$, let $[\![\phi]\!] = \{\Gamma \in \Omega_p[\psi] : \phi \in \Gamma\}$.

Lemma 4. *(1) $\Omega_p[\psi]$ is finite.*
(2) $2^{\Omega_p[\psi]} = \{[\![\phi]\!] : \phi \in \mathcal{L}[\psi]\}$.
(3) For any $\phi_1, \phi_2 \in \mathcal{L}[\psi]$, $\vdash \phi_1 \to \phi_2$ iff $[\![\phi_1]\!] \subseteq [\![\phi_2]\!]$.

Lemma 5. *For any $\Gamma \in \Omega_p[\psi]$, any $\phi \in \mathcal{L}[\psi]$ and any $\alpha \in A$ there exist*

$$x = max\{r \in \mathbb{Q}_p : L_r^\alpha \phi \in \Gamma\},\ y = min\{r \in \mathbb{Q}_p : \neg L_r^\alpha \phi \in \Gamma\}.$$

Moreover, $y = x + 1/p$.

Let Ω be the set of $\mathcal{L}(\mathcal{A})$-maximally consistent sets of formulas. We fix an injection $\pi : \Omega_p \to \Omega$ such that for any $\Gamma \in \Omega_p$, $\Gamma \subseteq \pi(\Gamma)$. We denote by $\Gamma^\infty = \pi(\Gamma)$, for any $\Gamma \in \Omega_p[\psi]$.

Lemma 6. *For any $\Gamma \in \Omega_p[\psi]$, any $\phi \in \mathcal{L}[\psi]$ and any $\alpha \in A$, there exists*

$$x^\infty = sup\{r \in \mathbb{Q} : L_r^\alpha \phi \in \Gamma^\infty\},\ y^\infty = inf\{r \in \mathbb{Q} : \neg L_r^\alpha \phi \in \Gamma^\infty\}.$$

Moreover, $x^\infty = y^\infty$ and $x \leq x^\infty < y$.

In what follows we denote by α_ϕ^Γ the value $x^\infty = y^\infty$ defined for $\phi \in \mathcal{L}[\psi]$, $\Gamma \in \Omega_q[\psi]$ and $\alpha \in A$.

Lemma 7. *$\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi)$ is an $\mathcal{A}$-CMP, where θ_ψ is defined for arbitrary $\Gamma \in \Omega_q[\psi]$ and $\alpha \in \mathcal{A}$ by*

$$\theta_\psi(\alpha)(\Gamma) = \begin{cases} D(\alpha_\phi^\Gamma, \Gamma) & \text{for } \alpha \in A \\ \omega & \text{for } \alpha \in \mathcal{A} \setminus A \end{cases}$$

Recall that $D(\alpha_\phi^\Gamma, \Gamma)$ is the α_ϕ^Γ-Dirac measure on Γ[5] *and ω is the null measure.*

Now we can prove the Truth Lemma.

Lemma 8 (Truth Lemma). *If $\phi \in \mathcal{L}[\psi]$, then $[\mathcal{M}_\psi, \Gamma \Vdash \phi$ iff $\phi \in \Gamma]$.*

Proof. Induction on the structure of ϕ. The only nontrivial case is $\phi = L_r^\alpha \phi'$.
($\Longrightarrow$) Suppose that $\mathcal{M}_\psi, \Gamma \Vdash L_r^\alpha \phi'$. Because $\phi \in \Gamma$, $r \in \mathbb{Q}_n \subseteq \mathbb{Q}_p$. Suppose $\phi \notin \Gamma$. Because $\phi \in \mathcal{L}[\psi]$ and Γ is $\mathcal{L}[\psi]$-maximally consistent, $\neg\phi \in \Gamma$. Let $y = min\{r \in \mathbb{Q}_p : \neg L_r^\alpha \phi \in \Gamma\}$. Then, from $\neg L_r^\alpha \phi' \in \Gamma$, we obtain $r \geq y$. But $\mathcal{M}_\psi, \Gamma \Vdash L_r^\alpha \phi'$ is equivalent with $\theta_\psi(\alpha)(\Gamma)([\![\phi']\!]) \geq r$, i.e. $\alpha_{\phi'}^\Gamma \geq r$. On the other hand, in Lemma 5 we proved that $\alpha_{\phi'}^\Gamma < y$ - contradiction.
($\Longleftarrow$) Suppose that $L_r^\alpha \phi' \in \Gamma$. Then $r \leq \alpha_\phi^\Gamma$, implying $\theta_\psi(\alpha)(\Gamma)([\![\phi]\!]) \geq r$. Hence, $\Gamma \Vdash L_r^\alpha \phi$.

[4] There might be more than one such projections.
[5] Notice that Γ is an element of a base of $2^{\Omega_p[\psi]}$ and $\alpha_\phi^\Gamma \in \mathbb{Q}_+$.

The previous lemma implies the small model property for our logic.

Theorem 4 (Small model property). *For any $\mathcal{L}(\mathcal{A})$-consistent formula ϕ, there exists an $\mathcal{A}$-CMP $\mathcal{M} = (M, \Sigma, \theta)$ with finite support and cardinality bound by the structure of ϕ, and $m \in M$ such that $\mathcal{M}, m \Vdash \phi$.*

The small model property proves the completeness of the axiomatic system, as already stated in Lemma 3.

Theorem 5 (Completeness). *CML is complete with respect to the semantics based on $\mathcal{A}$-CMPs, i.e. if $\Vdash \psi$, then $\vdash \psi$.*

7 Conclusions

In this paper we have introduced continuous Markovian logic for semantics based on continuous-space and continuous-time Markov processes. CML shares many characteristics with Hennessy-Milner logic, having operators indexed with transition labels and rates. We have proved that CML characterizes the stochastic bisimulation of CMPs, an ability shared with CSL logic. While CSL has operators that express probabilistic information per time intervals, CML plays directly on the level of the parameter of the exponential distribution that models the transition. In this way we obtain a very compact logic with a well-behaved semantics. We have identified a complete Hilbert-style axiomatization for CML and have proved the small model property. Questions concerning the decidability and complexity of model checking remain open. These we intend to address in future work.

We consider one of the major contributions of this paper to be highlighting the similarity between continuous and discrete time Markov processes (with continuous state-space) as mathematical structures: the rates of transitions behaving as the probabilities in the discrete models. Using this intuition, which has directly produced CML, we expect that it will be straightforward to adapt many results presented in [18] from discrete time to continuous Markov processes.

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Appendix

In this appendix we have collected the proofs of the major results and the detailed discussions of the examples presented in the paper.

Proof (Lemma 2). Before starting this proof, we introduce some additional concepts and present some results that are needed for our proof.

Given a set X, a family of subsets $\Pi \subset 2^X$ closed under finite intersection is called *π-system*. A family of subsets $\Lambda \subset 2^X$ is a *λ-system* if contains X and is closed under complementation and countable union of pairwise disjoint sets.

[Dynkin's $\lambda - \pi$ theorem]. *If Π is a π-system and Λ is a λ-system, then $\Pi \subset \Lambda$ implies $\overline{\Pi} \subseteq \Lambda$, where $\overline{\Pi}$ is the σ-algebra generated by Π.*

This theorem allows us to prove the next lemma.

[Lemma A.] *Suppose that $\Pi \subseteq 2^X$ is a π-system with $X \in \Pi$ and μ, ν are two measures on $(X, \overline{\Pi})$. If μ and ν agree on all the sets in Π, then they agree on $\overline{\Pi}$.*

We also present two more lemmas (see, e.g., [18] Section 7.7).

[Lemma B.] *Let (M, Σ) be an analytic space and let Σ_0 be a countably generated sub-σ-algebra of Σ which separates points in M, i.e., for any $m, n \in M$, $m \neq n$, there exists $S \in \Sigma_0$ such that $m \in S \not\ni n$. Then $\Sigma_0 = \Sigma$.*

[Lemma C.] *Let (M, Σ) be an analytic space and let $\equiv$ be an equivalence relation on M. If there exists a sequence $f_1, f_2, ...$ of real-valued Borel functions on M such that $m \equiv n$ iff for all i, $f_i(m) = f_i(n)$, then $(M^{\equiv}, \Sigma^{\equiv})$ is an analytic space.*

Now we prove Lemma 2.

For the beginning we show that $(M^{\approx}, \Sigma^{\approx})$ is an analytic space. Let $\mathcal{L}^*(\mathcal{A}) = \{\phi_i | i \in \mathbb{N}\}$. Because $[\![\phi_i]\!]_{\mathcal{M}}$ is measurable, the characteristic functions $1_{\phi_i} : M \to \{0, 1\}$ are measurable and $m \approx n$ iff $[\forall i \in \mathbb{N}, 1_{\phi_i}(m) = 1_{\phi_i}(n)]$. Lemma C proves further that $(M^{\approx}, \Sigma^{\approx})$ is an analytic space.

Let $\mathcal{B} = \{\pi([\![\phi_i]\!]_{\mathcal{M}} | i \in \mathbb{N}\}$. We show that $\overline{\mathcal{B}} = \Sigma^{\approx}$. Obviously, $\mathcal{B} \subseteq \Sigma^{\approx}$, because for any $\pi([\![\phi_i]\!]_{\mathcal{M}}) \in \mathcal{B}$, $\pi^{-1}(\pi([\![\phi_i]\!]_{\mathcal{M}})) \in \Sigma$. Notice that $\overline{\mathcal{B}}$ separates points in $M^{\approx}$: let $C, D \in M^{\approx}$, $C \neq D$ and let $m \in \pi^{-1}(C)$, $n \in \pi^{-1}(D)$; because $m \not\approx n$, there exists $\phi \in \mathcal{L}^*(\mathcal{A})$ such that $m \in [\![\phi]\!]_{\mathcal{M}} \not\ni n$. Hence, we can apply Lemma B and we obtain $\overline{\mathcal{B}} = \Sigma^{\approx}$.

Now we define $\theta^{\approx}$ such that π is a zigzag. Notice first that π is measurable and surjective by definition. For each $C \in \Sigma^{\approx}$ and $\alpha \in \mathcal{A}$, let $\theta^{\approx}(\alpha)(m^{\approx})(C) = \theta(\alpha)(m)(\pi^{-1}(C))$.

This definition is correct: let $m, n \in m^{\approx}$, we prove that $\theta(\alpha)(m)$ and $\theta(\alpha)(n)$ agree on $\Sigma^{\approx}$. We show first that they agree on $[\![\phi]\!]_{\mathcal{M}} \in \mathcal{B}$. Suppose that we have $\theta(\alpha)(m)([\![\phi]\!]_{\mathcal{M}}) < r < \theta(\alpha)([\![\phi]\!]_{\mathcal{M}})$. Then, $\mathcal{M}, m \Vdash \neg L_r^{\alpha}\phi$ while $\mathcal{M}, n \Vdash L_r^{\alpha}\phi$ - impossible. Because $\mathcal{B}$ is closed under finite intersection ($[\![\phi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} = [\![\phi \wedge \psi]\!]_{\mathcal{M}}$) and $M = [\![\top]\!]_{\mathcal{M}} \in \mathcal{B}$, we apply Lemma A and obtain that $\theta(\alpha)(m)$ and $\theta(\alpha)(n)$ agree on $\Sigma^{\approx}$.

Now we only need to prove that for any $\alpha \in \mathcal{A}$, $\theta^{\approx}(\alpha)$ is measurable. Let $C \in \Sigma^{\approx}$ and A a Borel set of $\mathbb{R}^+$. We have

$$(\theta^{\approx})^{-1}(\{\mu \in \Delta(M^{\approx}, \Sigma^{\approx}) | \mu(B) \in A\}) = \pi((\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\})).$$

But $\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\}$ is measurable in $\Delta(M, \Sigma)$ and because $\theta(\alpha)$ is measurable we obtain that $(\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\}) \in \Sigma$ implying $\pi((\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\})) \in \Sigma^{\approx}$.

Proof (Lemma 3). [$\Vdash \psi$ implies $\vdash \psi$] is equivalent with [$\nvdash \psi$ implies $\nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies that there exists a model $\mathcal{M}$ such that $\mathcal{M}, m \nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies the satisfiability of $\neg\psi$]. This last term is equivalent to our working hypothesis.

Proof (Lemma 5). The existence of x and y derives from the construction of $\Omega_p[\psi]$ and the Rules (R2), (R3).
Because Γ is consistent and $L_x^\alpha \phi, \neg L_y^\alpha \phi \in \Gamma$, $x \neq y$. If $x > y$, $L_x^\alpha \phi \in \Gamma$ entails (Axiom (A2)) $L_y^\alpha \phi \in \Gamma$, contradicting the consistency of Γ.
Hence, $x < y$. If $x + 1/p < y$, then $L_{x+1/p}^\alpha \phi \notin \Gamma$ (because $x < x + 1/p \in \mathbb{Q}_q$ and Γ is $\mathcal{L}_p$-maximally consistent), i.e. $\neg L_{x+1/p}^\alpha \phi \in \Gamma$ implying that $x + 1/p \geq y$ - contradiction.

Proof (Lemma 6). As before, the existence of sup and inf is guaranteed by the construction and the Rules (R2) and (R3).

Suppose that $x^\infty < y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty < r < y^\infty$. This implies that $\neg L_r^\alpha \phi \in \Gamma^\infty$ (from the definition of x^∞) and $L_r^\alpha \phi \in \Gamma^\infty$ (from the definition of y^∞) - impossible because Γ^∞ is consistent.

Suppose that $x^\infty > y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty > r > y^\infty$. As Γ^∞ is maximally consistent we have either $L_r^\alpha \phi \in \Gamma^\infty$ or $\neg L_r^\alpha \phi \in \Gamma^\infty$. The first case contradicts the definition of x^∞ while the second the definition of y^∞.

Obviously, $x \leq x^\infty \leq y$. We cannot have $x^\infty = y$ because else $L_{x^\infty}^\alpha \phi, \neg L_{x^\infty}^\alpha \phi \in \Gamma$ contradicting the consistency of Γ.

Proof (Lemma 7). This result is a direct consequence of the construction of $\mathcal{M}_\psi$. First notice that because the space is discrete, is Polish, hence analytic space.

The central problem is to prove that for arbitrary $\Gamma \in \Omega_p[\psi]$ and $\alpha \in A$, the function $\theta_\psi(\alpha)(\Gamma) : 2^{\Omega_p[\psi]} \to \mathbb{R}^+$ is well defined and a measure on $(\Omega_p[\psi], 2^{\Omega_p[\psi]})$. Further, because the space is discrete with finite support, we obtain that $\theta(\alpha) \in [\![\Omega_p[\psi] \to \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]\!]$ that concludes the proof.

$\theta_\psi(\alpha)(\Gamma)$ is well defined: suppose that for $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ we have $[\![\phi_1]\!] = [\![\phi_2]\!]$. Then, from Lemma 4, $\vdash \phi_1 \leftrightarrow \phi_2$ and from Rule (R1) $\vdash L_r^\alpha \phi_1 \leftrightarrow L_r^\alpha \phi_2$. This entails $\alpha_{\phi_1}^\Gamma = \alpha_{\phi_2}^\Gamma$ and guarantees that $\theta_\psi(\alpha)(\Gamma)$ is well defined.

Now we prove that $\theta_\psi(\alpha)(\Gamma)$ is a measure.

For showing $\theta_\psi(\alpha)(\Gamma)(\emptyset) = 0$, we show that for any $r > 0$, $\vdash \neg L_r^\alpha \bot$. This is sufficient, as Axiom (A1) guarantees that $\vdash L_0^\alpha \bot$ and $[\![\bot]\!] = \emptyset$. Suppose that there exists $r > 0$ such that $L_r^\alpha \bot$ is consistent. Let $\epsilon \in (0, r) \cap \mathbb{Q}$. Then Axiom (A2) gives $\vdash L_r^\alpha \bot \to L_\epsilon^\alpha \bot$. Hence, $\vdash L_r^\alpha \bot \to (L_r^\alpha(\bot \wedge \bot) \wedge L_\epsilon^\alpha(\bot \wedge \neg\bot))$ and applying the Axiom (A3), $\vdash L_r^\alpha \bot \to L_{r+\epsilon}^\alpha \bot$. Repeating this argument we can prove that $\vdash L_r^\alpha \bot \to L_s^\alpha \bot$ for any s and Rule (R3) confirms the inconsistency of $L_r^\alpha \bot$.

We show now that if $A, B \in 2^{\Omega_p[\psi]}$ with $A \cap B = \emptyset$, then $\theta_\psi(\alpha)(\Gamma)(A) + \theta_\psi(\alpha)(\Gamma)(B) = \theta_\psi(\alpha)(\Gamma)(A \cup B)$. Let $A = [\![\phi_1]\!]$, $B = [\![\phi_2]\!]$ with $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ and $\vdash \phi_1 \to \neg\phi_2$. Let $x_1 = \theta_\psi(\alpha)(\Gamma)(A)$, $x_2 = \theta_\psi(\alpha)(\Gamma)(B)$ and $x = \theta_\psi(\alpha)(\Gamma)(A \cup B)$. We prove that $x_1 + x_2 = x$.

Suppose that $x_1 + x_2 < x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x_1' + x_2' < x$, where $x_i' = x_i + \epsilon_i$ for $i = 1, 2$. But this implies that $L_{x_i'}^\alpha \phi_i \not\in \Gamma^\infty$ (from the definition of x_i), hence $\neg L_{x_i'}^\alpha \phi_i \in \Gamma^\infty$. Further, using Axiom (A4), we obtain $\neg L_{x_1'+x_2'}^\alpha(\phi_1 \vee \phi_2) \in \Gamma^\infty$, implying (from the definition of x) that $x_1' + x_2' \geq x$ - contradiction.

Suppose that $x_1 + x_2 > x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x_1'' + x_2'' > x$, where $x_i'' = x_i - \epsilon_i$ for $i = 1, 2$. But this implies (from the definition of x_i) that $L_{x_i''}^\alpha \phi_i \in \Gamma^\infty$. Further, Axiom (A3) gives $L_{x_1''+x_2''}^\alpha(\phi_1 \vee \phi_2) \in \Gamma^\infty$, i.e. $x_1'' + x_2'' \leq x$ - contradiction.