Continuous Markovian Logic - From Complete Axiomatization to the Metric Space of Formulas*

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Abstract

In this paper we study the Continuous Markovian Logic (CML), a multimodal logic that expresses quantitative and qualitative properties of continuous-space and continuous-time labelled Markov processes(CMPs). The modalities of CML approximates the rates of the exponentially distributed random variables that characterize the duration of labeled transitions. We propose a sound-complete Hilbert-style axiomatization for CML against the CMP-semantics and prove the small model property. It is known, from the similar case of probabilistic systems, that such a logic characterizes bisimulation and supports the definition of a distance between a model and a formula that quantifies the satisfiability relation; only that this distance is not always computable. We prove that in our case it can be approximated, within a given error, by using a distance between logical formulas that we define relying on the small model property of CML.

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1 Introduction

Many complex natural and man-made systems (e.g., biological, ecological, physical, social, financial, and computational) are modeled as stochastic processes in order to handle either a lack of knowledge or inherent randomness. These systems are frequently studied both in interaction with discrete systems, such as controllers, or with interactive environments having continuous behavior. This context has motivated research aiming to develop a general theory of systems that is able to uniformly treat discrete, continuous and hybrid interactive systems. Two of the central questions of this research are *"when do two systems behave similarly up to some quantifiable observation error?"* and *"is there any (algorithmic) technique to check whether two systems have similar behaviours?"*. Both these questions are also related to the problems of state reduction (collapsing a model to an equivalent reduced model) and discretization (reduce a continuous or hybrid system to an equivalent discrete one), which are cornerstones in the field of stochastic systems.

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In the case of probabilistic systems, *probabilistic bisimulation* [16] relates systems with identical probabilistic behaviours. Along this line, it has been shown that a probabilistic multimodal logic (PML) [15, 16, 1, 10, 12], with operators indexed by the probabilities of the labelled transitions, characterizes probabilistic bisimulation: the logical equivalence induced on probabilistic models coincides with their bisimulation [16, 18, 9]. But in spite of the elegant theories supporting it, the concept of bisimulation remains too strict for applications, as it reflects only identical behaviours. In modelling, the values of the parameters (rates or probabilities) are in most of the cases approximations and consequently, one is interested to know whether two processes that differ by a small amount in real-valued parameters show similar (not necessarily identical) behaviours. In such cases, instead of bisimulation relation, we need a metric to estimate the degree of similarity of two systems in terms of behaviours.

To solve this problem for probabilistic systems, a class of pseudometrics have been proposed [19, 5, 18], to quantify the similarity of processes: two processes are at distance zero iff they are bisimilar; otherwise, they are closer when they differ by a small amount in their probabilistic behaviours. Moreover, it has been shown that these metrics can be defined on top of PML [5, 18]. One can extend the satisfiability relation of PML, $P \Vdash \phi$, to a function d such that $d(P,\phi) \in [0,1]$ measures the degree of satisfiability between the process P and formula ϕ. d induces a distance D between processes by $D(P,P') = sup\{|d(P,\phi) - d(P',\phi)|, \phi \in \mathcal{L}\}$, where $\mathcal{L}$ is the set of logical formulas. Because $\mathcal{L}$ is infinite, it is not always possible to have an algorithm to compute D. Moreover, if P is infinite or extremely big, already evaluating $d(P,\phi)$ is problematic. Sometimes, for a given P and ϕ, approximation techniques such as statistical model checking [14, 20] can help to evaluate $d(P,\phi)$ within a given error.

Relying on the observation that PML plays a central role in all the above mentioned developments for the case of probabilistic models, in this paper we take the challenge of developing and studying the similar logic for the case of stochastic (Markovian) processes. Our models are *continuous-time and continuous-space labelled Markovian processes* (CMPs), similar to the models proposed in [9, 3]. CMPs generalize probabilistic models such as *labeled Markov processes* [18, 8, 4, 7] and *Harsanyi type spaces* [11, 17]. Our logic, called *continuous Markovian logic* (CML), involves modal operators indexed with transition labels a and positive rationals r: $L_r^a\phi$ states that the rate of the a-transitions from the current state to the set of states satisfying ϕ is *at least* r, and $M_r^a\phi$ states that the same rate is *at most* r.

In spite of their syntactic similarities, CML and PML are very different. While in the probabilistic case the two modal operators are dual, being related by the rule $M_r^a\phi = L_{1-r}^a\neg\phi$, in the stochastic case they are independent. And there exists no sound equivalence of type $\neg X_r^a\phi \leftrightarrow Y_s^a\neg\phi$ for $X, Y \in \{L, M\}$, that generate some kind of positive normal forms for CML formulas, because the rate of the transitions from a given state m to the set of states satisfying ϕ is not related to the rate of the transitions from m to the set of states satisfying $\neg\phi$. The differences are reflected in the sound-complete axiomatizations that we present both for CML and for its fragment without M_r^a-operators: many axioms of PML [12, 21, 10] are not sound for CMPs, such as $\vdash L_r^a\top$ or $\vdash L_r^a\phi \rightarrow \neg L_s^a\neg\phi$ for $r+s<1$. Also at the level of the small model property, which in the case of PML [21] relies on the fact that for a fixed integer q there exist a finite number of integers p such that $p/q \in [0,1]$, a series of nontrivial additional problems rise in the stochastic case.

The construction of a small model for a consistent CML-formula is the cornerstone of this paper supporting not only the completeness proofs, but also an approximation technique to evaluate the extension $d(P,\phi) \in [0,1]$ of the satisfiability relation for CML. Because CML is completely axiomatized against CMPs and it characterizes stochastic bisimulation, we can approach the bisimulation-distance problems, addressed semantically in the probabilistic case

[19, 5, 18], as syntactic problems centered on provability. Formally, we define the distance $\overline{d}(\phi,\psi) = sup\{|d(P,\phi) - d(P,\psi)|, P \in \mathfrak{P}\}$ induced by d over the space of logical formulas, where $\mathfrak{P}$ is the class of CMPs. In the context of the complete axiomatization, $\overline{d}$ measures the similarity between logical formulas in terms of provability: ϕ and ψ are close, if they are logically equivalent or can be both (or their negations) proved from the same hypothesis. And *strong robustness theorem* holds: $d(P,\phi) \leq d(P,\psi) + \overline{d}(\phi,\psi)$. But the problem here is that $\overline{d}$ is not always computable, as it quantifies over the entire class of CMPs (which are infinite processes). At this point, our technique for constructing a finite model for a consistent formula plays its role: we can approximate $\overline{d}(\phi,\psi)$ by $\widetilde{d}(\phi,\psi) = max\{|d(P,\phi) - d(P,\psi)|, P \in \Omega_p[\phi,\psi]\}$, where $\Omega_p[\phi,\psi]$ is the finite model (finite set of processes) constructed for $\phi \wedge \psi$ if it is consistent, or for $\neg(\phi \wedge \psi)$ otherwise, and $p \in \mathbb{N}$ is the parameter involved in the construction. This proves the *weak robustness theorem*: $d(P,\phi) \leq d(P,\psi) + \widetilde{d}(\phi,\psi) + 1/p$, which is very useful in applications where it is expensive to evaluate $d(P,\phi)$. In such a case, one can evaluate $d(P,\psi)$ for a certain formula ψ and, using the weak robustness theorem, obtain for free an evaluation of $d(P,\phi)$ for any formula ϕ. Of course, the accuracy of this approximation depends on how similar ϕ and ψ are from provability perspective which influences both the distance $\widetilde{d}(\phi,\psi)$ and the parameter p of the finite model construction.

To summarize, the achievements of this paper are as follows.

- We introduce Continuous Markovian Logic, a modal logic that expresses quantitative and qualitative properties of continuous Markov processes. CML is endowed with operators that approximate the labelled transition rates of CMPs and allows us to reason on approximated properties. This logic characterizes the stochastic bisimulation of CMPs.
- We present two sound-complete Hilbert-style axiomatizations, for CMP and for its M_r^a-free fragment. These are very different from the similar probabilistic case, due to the structural differences between probabilistic and stochastic models. In the stochastic context M_r^a and L_r^a are independent operators and this induces significant differences between the axiomatization of the entire CML and of its fragment without M_r^a operators.
- We prove the finite model properties for CML and its restricted fragment. The constructions of the finite models are novel in the way they exploit the granularity and the Archimedian properties of positive rationals.
- We define a distance between logical formulas that relates with the distance between a model and a formula proposed in the literature (for probabilistic systems) to prove the robustness theorems. The organization of the space of logical formulas as a metric space with a pseudometric sensitive to the axiomatization and provability is a novelty in the field of metric semantics. Also the robustness theorems, that use simultaneously quantifications on semantic and syntactic levels, are original results.
- We show that the complete axiomatization and the finite model construction can be used to approximate the syntactic distance $\overline{d}$. This idea opens new research perspectives on the direction of designing algorithms to estimate such distances within given errors.

The structure of the paper. The first section establishes the background and some preliminary concepts used in the paper. Section 3 introduces CMPs and their bisimulation. In Section 4 we define the logic CML and in Section 5 we present sound-complete axiomatizations for both CML and its M_r^a-free fragment proving, at the same time, the small model properties. Section 6 introduces the metric semantics and the results related to metrics and bisimulation. We also have a section with conclusions and future work and an Appendix with additional definitions and proofs.

2 Preliminary definitions

In this section we introduce some notations and establish the terminology used in the paper.

For arbitrary sets A, B, 2^A denotes the powerset of A and $[A \to B]$ the set of functions from A to B.

Given a set M, $\Sigma \subseteq 2^M$ that contains M and is closed under complement and countable union is a *σ-algebra* over M; (M, Σ) is a *measurable space* and the elements of Σ are *measurable sets*. $\Omega \subseteq 2^M$ is a *base for* Σ if Σ is the closure of Ω under complement and countable union; we write $\overline{\Omega} = \Sigma$.

Given a relation $\mathfrak{R} \subseteq M \times M$, $N \subseteq M$ is $\mathfrak{R}$-closed iff $\{m \in M \mid \exists n \in N, (m, n) \in \mathfrak{R}\} \subseteq N$. If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

A *measure* on (M, Σ) is a function $\mu : \Sigma \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$ and for $\{N_i | i \in I \subseteq \mathbb{N}\} \subseteq \Sigma$ with pairwise disjoint elements, $\mu(\bigcup_{i \in I} N_i) = \sum_{i \in I} \mu(N_i)$.

Let $\Delta(M, \Sigma)$ be the class of measures on (M, Σ). We organize it as a measurable space by considering the σ-algebra generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets

$$\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}.$$

Given two measurable spaces (M, Σ) and (N, Θ), a mapping $f : M \to N$ is *measurable* if for any $T \in \Theta$, $f^{-1}(T) \in \Sigma$. We use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Θ).

Given a set X, a *pseudometric on* X is a function $d : X \times X \to \mathbb{R}^+$ such that
1. $\forall x \in X$, $d(x, x) = 0$;
2. $\forall x, y \in X$, $d(x, y) = d(y, x)$;
3. $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(y, z)$.

d is *a metric on* X if, in addition, satisfies

4. $\forall x, y \in X$, if $d(x, y) = 0$, then $x = y$.

If d is a metric, then (X, d) is a *metric space*.

Given a pseudometric on X, one can define an equivalence on X by pairing the elements at distance zero.

Central for this paper is the notion of an *analytic set*. We only recall here the main definition and mention the properties of analytic sets used in our proves. For detailed discussion on this topic related to Markov processes, the reader is referred to [18] (Section 7.5) or to [6] (Section 4.4).

A metric space (M, d) is *complete* if every Cauchy sequence converges in M.

A *Polish space* is the topological space underlying a complete metric space with a countable dense subset. Note that any discrete space is Polish.

An *analytic set* is the image of a Polish space under a continuous function between Polish spaces. Note that any Polish space is an analytic set.

There are some basic facts about analytic sets that we use in this paper. Firstly, an analytic set, as measurable space, has a denumerable base with disjoint elements. Secondly, If $\mathcal{M}_1, \mathcal{M}_2$ are analytic sets with Σ_1, Σ_2 the Borel algebras generated by their topologies, then the product space $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with the Borel algebra Σ generated by the product topology is an analytic set.

3 Continuous Markov processes

Based on an equivalence between the definitions of Harsanyi type spaces [11, 17] and labelled Markov processes [18, 8, 4, 7] evidenced by Doberkat [6], we introduce the *continuous Markov* *processes* (CMPs), models of stochastic systems with continuous state space and continuous time transitions. CMPs are defined for a fixed countable set $\mathcal{A}$ of *transition labels* which represent types of interactions with the environment. If $a \in \mathcal{A}$, m is the current state of the system and N is a measurable set of states, the function $\theta(\alpha)(m)$ is a measure on the state space and $\theta(\alpha)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable that characterizes the duration of an α-transition from m to arbitrary $n \in N$. Indeterminacy in such systems is resolved by races between events executing at different rates.

▶ **Definition 1** (Continuous Markov processes)**.** Given an analytic set (M, Σ), where Σ is the Borel algebra generated by the topology, an $\mathcal{A}$-*continuous Markov kernel* is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where $\theta : \mathcal{A} \to [\![M \to \Delta(M, \Sigma)]\!]$. M is the support set of $\mathcal{M}$ denoted by $sup(\mathcal{M})$. If $m \in M$, $(\mathcal{M}, m)$ is an $\mathcal{A}$-*continuous Markov process*[1].

In the rest of the paper we assume that the set of transition labels $\mathcal{A}$ is fixed. We denote by $\mathfrak{M}$ the class of $\mathcal{A}$-continuous Markov kernels (CMKs) and use $\mathcal{M}, \mathcal{M}_i, \mathcal{M}'$ to range over $\mathcal{M}$. We denote by $\mathfrak{P}$ the set of $\mathcal{A}$-CMPs and use P, P_i, P' to range over $\mathfrak{P}$.

The stochastic bisimulation for CMPs follows the line of the Larsen-Skou probabilistic bisimulation [16, 7, 18].

▶ **Definition 2** (Stochastic Bisimulation)**.** Given $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, a *rate-bisimulation relation* on $\mathcal{M}$ is a relation $\mathfrak{R} \subseteq M \times M$ such that $(m, n) \in \mathfrak{R}$ iff for any $C \in \Sigma(\mathfrak{R})$ and any $a \in \mathcal{A}$,

$$\theta(a)(m)(C) = \theta(a)(n)(C).$$

Two processes $(\mathcal{M}, m)$ and $(\mathcal{M}, n)$ are *stochastic bisimilar*, written $m \sim_{\mathcal{M}} n$, if they are related by a rate-bisimulation relation.

Observe that, for any $\mathcal{M} \in \mathfrak{M}$ there exist rate-bisimulation relations. For instance, the identity of the elements of the support-set of $\mathcal{M}$ is a rate-bisimulation relation. The stochastic bisimulation of processes from different CMKs is defined by considering the disjoint union of CMKs.

If $\mathcal{M} = (M, \Sigma, \theta), \mathcal{M}' = (M', \Sigma', \theta') \in \mathfrak{M}$, then $\mathcal{M}'' = (M'', \Sigma'', \theta'') \in \mathfrak{M}$ is the *disjoint union of* $\mathcal{M}$ *and* $\mathcal{M}'$ if $M'' = M \uplus M'$, $\Sigma'' = \overline{\Sigma \uplus \Sigma'}$ and for any $a \in \mathcal{A}$, $N \in \Sigma$ and $N' \in \Sigma'$,

$$\theta''(a)(m)(N \uplus N') = \begin{cases} \theta(a)(m)(N) & \text{if } m \in M \\ \theta'(a)(m)(N') & \text{if } m \in M' \end{cases}$$

We denote $\mathcal{M}''$ by $\mathcal{M} \uplus \mathcal{M}'$. If $m \in M$ and $m' \in M'$, we say that $(\mathcal{M}, m)$ and $(\mathcal{M}', m')$ are *bisimilar* written $(\mathcal{M}, m) \sim (\mathcal{M}', m')$ whenever $m \sim_{\mathcal{M} \uplus \mathcal{M}'} m'$.

4 Continuous Markovian Logics

In this section we introduce the continuous Markovian logic (CML) for semantics based on CMPs. In addition to the Boolean operators, this logic is endowed with *stochastic modal operators* that approximate the rates of transitions. Thus, for $a \in \mathcal{A}$ and $r \in \mathbb{Q}_+$, $L_r^a \phi$ characterizes a CMP $(\mathcal{M}, m)$ such that the rate of the a-transition from m to the class of the

[1] $\theta(a)$ is a measurable mapping between (M, Σ) and $\Delta(M, \Sigma)$. This is equivalent with the conditions on the two-variable *rate function* used in [9] to define continuous Markov processes; for the proof of the equivalence see, e.g. Proposition 2.9, of [6].

states characterized by ϕ is *at least* r; symmetrically, $M_r^a\phi$ is satisfied when the same rate is *at most* r. In this respect, this logic is similar to probabilistic logics such as [1, 15, 12, 21, 10]. CMLs extends these logics to stochastic domains. The obvious structural similarities between the probabilistic and the stochastic models are not preserved when we consider the logic. By focusing on general measures instead of probabilistic measures in the definition of the transition systems, many of the axioms of probabilistic logics, presented e.g., in [12, 21, 10], are not sound for stochastic semantics. This is the case with $\vdash L_r^a\top$ or $\vdash L_r^a\phi \to \neg L_s^a\neg\phi$ for $r+s<1$. Moreover, while in probabilistic settings the operators L_r^a and M_s^a are dual, satisfying $M_r^a\phi = L_{1-r}^a\neg\phi$, they became independent in stochastic semantics. For this reason, in the next section we study two CML logics with complete axiomatizations, $\mathcal{L}(\mathcal{A})$ involving only the stochastic operators of type L_r^a, and $\mathcal{L}^+(\mathcal{A})$ that comprises both L_r^a and M_s^a.

▶ **Definition 3** (Syntax)**.** Given a countable set $\mathcal{A}$, the formulas of $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^+(\mathcal{A})$ respectively are introduced by the following grammars, for arbitrary $a \in \mathcal{A}$ and $r \in \mathbb{Q}_+$.

$$\mathcal{L}(\mathcal{A}): \quad \phi := \top | \neg\phi \mid \phi\wedge\phi \mid L_r^a\phi,$$

$$\mathcal{L}^+(\mathcal{A}): \quad \phi := \top | \neg\phi \mid \phi\wedge\phi \mid L_r^a\phi \mid M_r^a\phi.$$

In what follows we use the same set $\mathcal{A}$ of labels that we have considered in the previous section in the definition of CMPs.

The semantics of $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^+(\mathcal{A})$, called in this paper *Markovian semantics*, are defined by the *satisfiability relation* for arbitrary $\mathcal{A}$-CMPs $(\mathcal{M}, m)$ with $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, by:

$\mathcal{M}, m \Vdash \top$ always,

$\mathcal{M}, m \Vdash \neg\phi$ iff it is not the case that $\mathcal{M}, m \Vdash \phi$,

$\mathcal{M}, m \Vdash \phi\wedge\psi$ iff $\mathcal{M}, m \Vdash \phi$ and $\mathcal{M}, m \Vdash \psi$,

$\mathcal{M}, m \Vdash L_r^a\phi$ iff $\theta(a)(m)([\![\phi]\!]_{\mathcal{M}}) \geq r$,

$\mathcal{M}, m \Vdash M_r^a\phi$ iff $\theta(a)(m)([\![\phi]\!]_{\mathcal{M}}) \leq r$,

where $[\![\phi]\!]_{\mathcal{M}} = \{m \in M | \mathcal{M}, m \Vdash \phi\}$.

Notice that the semantics of $L_r^a\phi$ and $M_r^a\phi$ are well defined only if $[\![\phi]\!]_{\mathcal{M}}$ is measurable. This is guaranteed by the fact that $\theta(a)$ is a measurable mapping between (M, Σ) and $\Delta(M, \Sigma)$, as proved in the next lemma.

▶ **Lemma 4.** *For any $\phi \in \mathcal{L}^+(\mathcal{A})$ and any $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$, $[\![\phi]\!]_{\mathcal{M}} \in \Sigma$.*

Proof. Induction on the structure of ϕ. The only nontrivial cases involve the stochastic operators. For $\phi = L_r^a\psi$, observe that

$$[\![L_r^a\psi]\!]_{\mathcal{M}} = (\theta(a))^{-1}(\{\mu \in \Delta(M, \Sigma) | \mu([\![\psi]\!]_{\mathcal{M}}) \geq r\}).$$

From the inductive hypothesis, $[\![\psi]\!]_{\mathcal{M}} \in \Sigma$, hence, $\{\mu \in \Delta(M, \Sigma) | \mu([\![\psi]\!]_{\mathcal{M}}) \geq r\}$ is measurable in $\Delta(M, \Sigma)$ and because $\theta(a)$ is a measurable mapping, we obtain that $[\![L_r^a\psi]\!]_{\mathcal{M}}$ is measurable. Similarly it can be proved for $\phi = M_r^a\psi$, because the σ-algebra generated by $\{\mu \in \Delta(M, \Sigma) | \mu([\![\psi]\!]_{\mathcal{M}}) \geq r\}$ on $\Delta(M, \Sigma)$ coincides with the σ-algebra generated by $\{\mu \in \Delta(M, \Sigma) | \mu([\![\psi]\!]_{\mathcal{M}}) \leq r\}$. ◀

When it is not the case that $\mathcal{M}, m \Vdash \phi$, we write $\mathcal{M}, m \nVdash \phi$.

We also consider $\bot = \neg\top$. Notice that always $\mathcal{M}, m \nVdash \bot$.

In $\mathcal{L}^+(\mathcal{A})$ we can define a derived operator $E_r^a\phi = L_r^a\phi \wedge M_r^a\phi$ with the semantics

$\mathcal{M}, m \Vdash E_r^a\phi$ iff $\theta(a)(m)([\![\phi]\!]_{\mathcal{M}}) = r$.

A formula ϕ is *satisfiable* if there exists $\mathcal{M} = (M, \Sigma, \theta) \in \mathfrak{M}$ and $m \in M$ such that $\mathcal{M}, m \Vdash \phi$. ϕ is *valid*, denoted by $\Vdash \phi$, if $\neg\phi$ is not satisfiable.

Notice that $E^a_r\phi$ characterizes the process that can do an a-transition to the set of processes satisfying ϕ with the rate r. So, in this case one can express the exact rate of the transitions. This is always possible in probabilistic logic where M^a_r and L^a_r are dual operators and consequently E^a_r is always definable. In the stochastic case L^a_r, M^a_r and E^a_r are mutually independent. We chose not to study a Markovian logic that involves only the E^a_r operator because this is not useful in applications where we do not know, in general, the exact rates of the transitions. It is instead more useful to work with approximations such as M^a_r or L^a_r.

5 Complete axiomatizations

In this section we present two Hilbert-style axiomatizations, one for $\mathcal{L}(\mathcal{A})$ and one for $\mathcal{L}^+(\mathcal{A})$, and we prove that they are sound-complete against the Markovian semantics. Both axiomatizations, as in the case of the axiomatization proposed in [21] for probabilistic systems, contain infinitary rules that encode the Archimedean properties of $\mathbb{Q}_+ \cup \{+\infty\}$. However, as it has been shown in [13] following the line of [12], a finitary axiomatic system can be given by replacing the stochastic operators with some more complex operators. For our purpose, which is reasoning on approximated properties of Markovian processes, a complete axiomatization with simple axioms involving only the stochastic operators introduced before (and the Archimedian rules) is more useful. This is because in real applications we expect to use $M^a_{p/n}$ and $L^a_{p/n}$ for a fixed integer n ($1/n$ is the modelling error). In this context, the Archimedian rules will never be effectively used, but they will only guarantee that once we have established that $(\mathcal{M}, m) \Vdash L^a_{p/n}\phi \wedge M^a_{(p+1)/n}\phi$, this remains true and converges when we use better and better approximations of the model.

As usual, we say that a formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proved from the given axioms and rules. We say that ϕ is *consistent*, if $\phi \to \bot$ is not provable. Given a set Φ of formulas, we say that Φ proves ϕ, $\Phi \vdash \phi$, if from the formulas of Φ and the axioms one can prove ϕ. Φ is *consistent* if it is not the case that $\Phi \vdash \bot$. If Φ is finite we denote by $\bigwedge \Phi = \bigwedge_{\phi \in \Phi} \phi$. For a sublanguage $\mathcal{L} \subseteq \mathcal{L}^+(\mathcal{A})$, we say that Φ is $\mathcal{L}$*-maximally consistent* if Φ is consistent and no formula of $\mathcal{L}$ can be added to it without making it inconsistent.

5.1 Axiomatization for $\mathcal{L}(\mathcal{A})$

Table 1 contains a Hilbert-style axiomatization for $\mathcal{L}(\mathcal{A})$. The axioms and rules[2] are given for propositional variables $\phi, \psi \in \mathcal{L}(\mathcal{A})$, for arbitrary $a \in \mathcal{A}$ and $s, r \in \mathbb{Q}^+$.

(A1):	$\vdash L^a_0\phi$
(A2):	$\vdash L^a_{r+s}\phi \to L^a_r\phi$
(A3):	$\vdash L^a_r(\phi \wedge \psi) \wedge L^a_s(\phi \wedge \neg\psi) \to L^a_{r+s}\phi$
(A4):	$\vdash \neg L^a_r(\phi \wedge \psi) \wedge \neg L^a_s(\phi \wedge \neg\psi) \to \neg L^a_{r+s}\phi$
(R1):	If $\vdash \phi \to \psi$ then $\vdash L^a_r\phi \to L^a_r\psi$
(R2):	If $\forall r < s, \vdash \phi \to L^a_r\psi$ then $\vdash \phi \to L^a_s\psi$
(R3):	If $\forall r > s, \vdash \phi \to L^a_r\psi$ then $\vdash \phi \to \bot$

Table 1 The axiomatic system of $\mathcal{L}(\mathcal{A})$

[2] These are considered in addition to the axiomatization of propositional logic.

This axiomatic system has some similarities to the axiomatic system of probabilistic logic proposed in [21] for Harsanyi type spaces. The main difference is that the axioms of probabilistic logic $\vdash L_r^a\top$ and $\vdash L_r^a\phi \to \neg L_s^a\neg\phi$ for $r+s \leq 1$ are not sound for the Markovian semantics and this changes the entire proof structure. We also have two Archimedean properties reflected in (R2) and (R3); while the first allows us to argue on convergent sequences of rationals, the second excludes the models with infinite rates.

▶ **Theorem 5** (Soundness). *The axiomatic system of $\mathcal{L}(\mathcal{A})$ is sound for the Markovian semantics, i.e., for any $\phi \in \mathcal{L}(\mathcal{A})$, if $\vdash \phi$ then $\Vdash \phi$.*

In what follows we prove the finite model property for $\mathcal{L}(\mathcal{A})$ using the filtration method adapted for CMPs. This result will eventually establish the (weak) completeness of the axiomatic system for the Markovian semantics, meaning that everything that is true for all the models is also provable. Formally, for an arbitrary consistent formula $\psi \in \mathcal{L}(\mathcal{A})$, we will construct a CMP $(\mathcal{M}_\psi, \Gamma)$ where $sup(\mathcal{M}_\psi)$ is a finite set of $\mathcal{L}(\mathcal{A})$-consistent sets of formulas. As usual with the filtration method, the key argument is the truth lemma: $\psi \in \Gamma$ iff $\mathcal{M}_\psi, \Gamma \Vdash \psi$. A similar construction has been proposed in [21] for probabilistic logic, where the finite model property derives from the fact that the number of rationals of type $\frac{p}{n}$, for a fixed integer n, is finite within $[0, 1]$. The same property does not hold in our case, as the focus is on $[0, \infty)$, and instead we need a more complicated construction.

Before proceeding with the construction, we fix some notations.

For $n \in \mathbb{N}$, $n \neq 0$, let $\mathbb{Q}_n = \{\frac{p}{n} : p \in \mathbb{N}\}$. If $S \subseteq \mathbb{Q}$ is finite, the *granularity of* S, $gr(S)$, is the lowest common denominator of the elements of S.
The *modal depth of* $\phi \in \mathcal{L}(\mathcal{A})$ is defined by $md(\top) = 0$, $md(\neg\phi) = md(\phi)$, $md(\phi \wedge \psi) = max(md(\phi), md(\psi))$ and $md(L_r^a\phi) = md(\phi) + 1$.
The *granularity of* $\phi \in \mathcal{L}$ is $gr(\phi) = gr(R)$, where $R \subseteq \mathbb{Q}_+$ is the set of indexes r of the operators L_r^a present in ϕ; the *upper bound of* ϕ is $max(\phi) = max(R)$.
The *actions of* ϕ is the set $act(\phi) \subseteq \mathcal{A}$ of indexes $a \in \mathcal{A}$ of the operators L_r^a present in ϕ.
For arbitrary $n \in \mathbb{N}$ and $A \subseteq \mathcal{A}$, let $\mathcal{L}_n(A)$ be the sublanguage of $\mathcal{L}(\mathcal{A})$ that uses only modal operators L_r^a with $r \in \mathbb{Q}_n$ and $a \in A$. For $\Lambda \subseteq \mathcal{L}(\mathcal{A})$, let $[\Lambda]_n = \{\phi \in \mathcal{L}_n(\mathcal{A}) : \Lambda \vdash \phi\}$.

Consider a consistent formula $\psi \in \mathcal{L}(\mathcal{A})$ with $gr(\psi) = n$ and $act(\psi) = A$.
Let $\mathcal{L}[\psi] = \{\phi \in \mathcal{L}_n(A) \mid max(\phi) \leq max(\psi), md(\phi) \leq md(\psi)\}$.

In what follows we construct $\mathcal{M}_\psi \in \mathfrak{M}$ such that each $\Gamma \in \text{sup}(\mathcal{M}_\psi)$ is a consistent set of formulas that contains an $\mathcal{L}[\psi]$-maximally consistent set of formulas and each $\mathcal{L}[\psi]$-maximally consistent set is contained in some $\Gamma \in \text{sup}(\mathcal{M}_\psi)$. And we will prove that for $\phi \in \mathcal{L}[\psi]$, $\phi \in \Gamma$ iff $\mathcal{M}_\psi, \Gamma \Vdash \phi$.

Let $\Omega[\psi]$ be the set of $\mathcal{L}[\psi]$-maximally consistent sets of formulas. $\Omega[\psi]$ is finite and any $\Lambda \in \Omega[\psi]$ contains finitely many *nontrivial formulas*[3]; in the rest of this construction we only count non-trivial formulas while ignoring the rest and when we use $\bigwedge\Lambda$ we refer to the conjunction of the nontrivial ones.

For each $\Lambda \in \Omega[\psi]$, such that $\{\phi_1, ..., \phi_i\} \subseteq \Lambda$ is its set of its non-trivial formulas, we construct $\Lambda^+ \supseteq [\Lambda]_n$ with the property that $\forall\phi \in \Lambda$ and $a \in \mathcal{A}$ there exists $\neg L_r^a\phi \in \Lambda^+$.

The construction step [ϕ_1 versus Λ:]
(R3) guarantees that $\exists r \in \mathbb{Q}_n$ s.t. $[\Lambda]_n \cup \{\neg L_r^a\phi_1\}$ is consistent (suppose that this is not the case, then $\vdash \bigwedge\Lambda \to L_r^a\phi_1$ for all $r \in \mathbb{Q}_n$ implying that $\bigwedge\Lambda$ inconsistent - impossible).

[3] By nontrivial formulas we mean the formulas that are not obtained from more basic consistent ones by boolean derivations.

Let $y_1^a = min\{s \in \mathbb{Q}_n : [\Lambda]_n \cup \{\neg L_s^a \phi_1\}$ is consistent$\}$ and $x_1^a = max\{s \in \mathbb{Q}_n : L_s^a \phi_1 \in [\Lambda]_n\}$ ((R3) guarantees the existence of max, because otherwise $\vdash \bigwedge \Lambda \to L_r^a \phi_1$ for all r implying $\bigwedge \Lambda$ inconsistent - impossible). (R2) implies that $\exists r \in \mathbb{Q} \setminus \mathbb{Q}_n$ s.t., $x_1^a < r < y_1^a$ and $\{\neg L_r^a \phi_1\} \cup [\Lambda]_n$ is consistent (otherwise, $\vdash \bigwedge \Lambda \to L_r^a \phi_1$ for all $r < y_1^a$ and due to (R2), $\vdash \bigwedge \Lambda \to L_{y_1^a}^a \phi_1$ - contradiction with the consistency of Λ). Obviously, $r \not\in \mathbb{Q}_n$. Let $n_1 = gran\{1/n, r\}$. Let $s_1^a = min\{s \in \mathbb{Q}_{n_1} : [\Lambda]_{n_1} \cup \{\neg L_s^a \phi_1\}$ is consistent$\}$, $\Lambda_1^a = \Lambda \cup \{\neg L_{s_1^a}^a \phi_1\}$ and $\Lambda_1 = \bigcup_{a \in A} \Lambda_1^a$.

The construction step [ϕ_2 versus Λ_1:]
As before, let $y_2^a = min\{s \in \mathbb{Q}_{n_1} : [\Lambda_1]_{n_1} \cup \{\neg L_s^a \phi_2\}$ is consistent$\}$ and $x_2^a = max\{s \in \mathbb{Q}_{n_1} : L_s^a \phi_2 \in [\Lambda_1]_{n_1}\}$. There exists $r \in \mathbb{Q} \setminus \mathbb{Q}_{n_1}$ s.t., $x_2^a < r < y_2^a$ and $\{\neg L_r^a \phi_2\} \cup [\Lambda_1]_{n_1}$ is consistent. Let $n_2 = gran\{1/n_1, r\}$. Let $s_2^a = min\{s \in \mathbb{Q}_{n_2} : [\Lambda]_{n_2} \cup \{\neg L_s^a \phi_2\}$ is consistent$\}$, $\Lambda_2^a = \Lambda_1 \cup \{\neg L_{s_2^a}^a \phi_2\}$ and $\Lambda_2 = \bigcup_{a \in A} \Lambda_2^a$.

We repeat this construction step for [ϕ_3 versus Λ_2],..,[ϕ_i versus Λ_{i-1}] and in a finite number of steps we eventually obtain $\Lambda \subseteq \Lambda_1 \subseteq ... \subseteq \Lambda_i$, where Λ_i is a consistent set containing a finite set of nontrivial formulas. Let $n_\Lambda = gran\{1/n_1, .., 1/n_i\}$. We make this construction for all $\Lambda \in \Omega[\psi]$. Let $p = gran\{1/n_\Lambda : \Lambda \in \Omega[\psi]\}$. Notice that $p > n$. Let $\Lambda^+ = [\Lambda_i]_p$ and $\Omega^+[\psi] = \{\Lambda^+ : \Lambda \in \Omega[\psi]\}$.

▶ Remark. Any consistent formula $\phi \in \mathcal{L}[\psi]$ is an element of a set $\Lambda^+ \in \Omega^+[\psi]$. For each $\Lambda \in \Omega[\psi]$, each $\phi \in \Lambda$ and $a \in \mathcal{A}$, there exist $s, t \in \mathbb{Q}_p$, $s < t$, such that $L_s^a \phi, \neg L_t^a \phi \in \Lambda^+$. Moreover, for any Λ^+ there exists a formula ρ such that $\phi \in \Lambda^+$ iff $\vdash \rho \to \phi$.

Let Ω_p be the set of $\mathcal{L}_p(A)$-maximally consistent sets of formulas. We fix an injective function[4] $f : \Omega^+[\psi] \to \Omega_p$ such that for any $\Lambda^+ \in \Omega^+[\psi]$, $\Lambda^+ \subseteq f(\Lambda^+)$. We denote by $\Omega_p[\psi] = f(\Omega^+[\psi])$. For $\phi \in \mathcal{L}[\psi]$, let $[\![\phi]\!] = \{\Gamma \in \Omega_p[\psi] : \phi \in \Gamma\}$. Anticipating the further construction, we will use $\Omega_p[\psi]$ as the support-set for $\mathcal{M}_\psi$. For this reason we establish some properties for this set.

▶ **Lemma 6.** *1.* $\Omega_p[\psi]$ *is finite.*
2. $2^{\Omega_p[\psi]} = \{[\![\phi]\!] : \phi \in \mathcal{L}[\psi]\}$.
3. *For any* $\phi_1, \phi_2 \in \mathcal{L}[\psi]$, $\vdash \phi_1 \to \phi_2$ *iff* $[\![\phi_1]\!] \subseteq [\![\phi_2]\!]$.
4. *For any* $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}[\psi]$ *and* $a \in A$ *there exist* $x = max\{r \in \mathbb{Q}_p : L_r^a \phi \in \Gamma\}$, $y = min\{r \in \mathbb{Q}_p : \neg L_r^a \phi \in \Gamma\}$ *and* $y = x + 1/p$.

Let Ω be the set of $\mathcal{L}(\mathcal{A})$-maximally consistent sets of formulas. We fix an injection $g : \Omega_p \to \Omega$ such that for any $\Gamma \in \Omega_p$, $\Gamma \subseteq \pi(\Gamma)$. We denote by $\Gamma^\infty = g(\Gamma)$, for any $\Gamma \in \Omega_p[\psi]$.

▶ **Lemma 7.** *For any* $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}[\psi]$ *and* $a \in A$, *there exists*

$$z = sup\{r \in \mathbb{Q} : L_r^a \phi \in \Gamma^\infty\} = inf\{r \in \mathbb{Q} : \neg L_r^a \phi \in \Gamma^\infty\} \text{ and } x \le z < y.$$

We denote z by a_ϕ^Γ and now we can define $\mathcal{M}_\psi$.

▶ **Lemma 8.** *If* $\theta_\psi : \mathcal{A} \to [\Omega_p[\psi] \to \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]$ *is defined for arbitrary* $a \in \mathcal{A}$, $\Gamma \in \Omega_q[\psi]$ *and* $\phi \in \mathcal{L}[\psi]$ *by* $\theta_\psi(a)(\Gamma)([\![\phi]\!]) = a_\phi^\Gamma$, *then* $\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi) \in \mathfrak{M}$.

Now we can prove the Truth Lemma.

▶ **Lemma 9** (Truth Lemma)**.** *If* $\phi \in \mathcal{L}[\psi]$, *then* $[\mathcal{M}_\psi, \Gamma \Vdash \phi$ *iff* $\phi \in \Gamma]$.

[4] This function is not unique.

Proof. Induction on the structure of ϕ. The only nontrivial case is $\phi = L_r^a\phi'$.
($\Longrightarrow$) Suppose that $\mathcal{M}_\psi, \Gamma \Vdash \phi$ and $\phi \notin \Gamma$. Hence $\neg\phi \in \Gamma$. Let $y = min\{r \in \mathbb{Q}_p : \neg L_r^a\phi \in \Gamma\}$. Then, from $\neg L_r^a\phi' \in \Gamma$, we obtain $r \geq y$. But $\mathcal{M}_\psi, \Gamma \Vdash L_r^a\phi'$ is equivalent with $\theta_\psi(a)(\Gamma)([\![\phi']\!]) \geq r$, i.e. $a_{\phi'}^\Gamma \geq r$. On the other hand, from Lemma 6, $a_{\phi'}^\Gamma < y$ - contradiction.
($\Longleftarrow$) If $L_r^a\phi' \in \Gamma$, then $r \leq a_\phi^\Gamma$ and $r \leq \theta_\psi(a)(\Gamma)([\![\phi]\!])$. Hence, $\mathcal{M}_\psi, \Gamma \Vdash L_r^a\phi$. ◀

The previous lemma implies the small model property for our logic.

▶ **Theorem 10** (Small model property). *For any $\mathcal{L}(\mathcal{A})$-consistent formula ϕ, there exists $\mathcal{M} \in \mathfrak{M}$ with finite support of cardinality bound by the structure of ϕ, and there exists $m \in sup(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$.*

The small model property proves the (weak) completeness of the axiomatic system.

▶ **Theorem 11** (Completeness). *The axiomatic system of $\mathcal{L}(\mathcal{A})$ is complete with respect to the Markovian semantics, i.e. if $\Vdash \psi$, then $\vdash \psi$.*

Proof. We have that [$\Vdash \psi$ implies $\vdash \psi$] is equivalent with [$\nvdash \psi$ implies $\nVdash \psi$], that is equivalent with [the consistency of $\neg\psi$ implies the existence of a model $(\mathcal{M}, m)$ for ψ] and this is guaranteed by the finite model property. ◀

5.2 Axiomatization for $\mathcal{L}^+(\mathcal{A})$

Table 2 contains a Hilbert-style axiomatization for $\mathcal{L}^+(\mathcal{A})$.

(B1):	$\vdash L_0^a\phi$
(B2):	$\vdash L_{r+s}^a\phi \to \neg M_r^a\phi,\ s > 0$
(B3):	$\vdash \neg L_r^a\phi \to M_r^a\phi$
(B4):	$\vdash \neg L_r^a(\phi \wedge \psi) \wedge \neg L_s^a(\phi \wedge \neg\psi) \to \neg L_{r+s}^a\phi$
(B5):	$\vdash \neg M_r^a(\phi \wedge \psi) \wedge \neg M_s^a(\phi \wedge \neg\psi) \to \neg M_{r+s}^a\phi$
(S1):	If $\vdash \phi \to \psi$ then $\vdash L_r^a\phi \to L_r^a\psi$
(S2):	If $\forall r < s, \vdash \phi \to L_r^a\psi$ then $\vdash \phi \to L_s^a\psi$
(S3):	If $\forall r > s, \vdash \phi \to M_r^a\psi$ then $\vdash \phi \to M_s^a\psi$
(S4):	If $\forall r > s, \vdash \phi \to L_r^a\psi$ then $\vdash \phi \to \bot$

Table 2 The axiomatic system of $\mathcal{L}^+(\mathcal{A})$

Notice the differences between these axioms and the axioms in Table 1. First of all the axiom (A2) had to be enforced by (B2) and (B3) which depict the connection between the two stochastic operators. In the probabilistic case these relations are encoded by the duality rule $M_r^a\phi = L_{1-r}^a\neg\phi$ and by the axiom $\vdash L_r^a\phi \to \neg L_s^a\neg\phi$ for $r + s < 1$; these two are not sound for stochastic models. Rule (A3) has been itself enforced by (B5). We also have an extra Archimedian rule for M_r^a. We prove below that all the theorems of $\mathcal{L}(A)$ are also theorems of $\mathcal{L}^+(\mathcal{A})$ and we state some theorems of $\mathcal{L}^+(\mathcal{A})$ that are central for the completeness proof.

▶ **Lemma 12.** *1.* $\vdash L_{r+s}^a\phi \to L_r^a\phi$, *2.* $\vdash M_r^a\phi \to M_{r+s}^a\phi$,
3. $\vdash L_r^a(\phi \wedge \psi) \wedge L_s^a(\phi \wedge \neg\psi) \to L_{r+s}^a\phi$, *4.* $\vdash M_r^a(\phi \wedge \psi) \wedge M_s^a(\phi \wedge \neg\psi) \to M_{r+s}^a\phi$,
5. $\vdash \neg M_r^a\phi \to L_r^a\phi$, *6.* $\vdash M_r^a\phi \to \neg L_{r+s}^a\phi,\ s > 0$,
7. If $\vdash \phi \to \psi$, *then* $\vdash M_r^a\psi \to M_r^a\phi$.

▶ **Theorem 13** (Soundness). *The axiomatic system of $\mathcal{L}^+(\mathcal{A})$ is sound for the Markovian semantics, i.e., for any $\phi \in \mathcal{L}^+(\mathcal{A})$, if $\vdash \phi$ then $\Vdash \phi$.*

The finite model property for $\mathcal{L}^+(\mathcal{A})$ is proved, similarly to the case of $\mathcal{L}(\mathcal{A})$, by using the filtration method. In what follows we will not reproduce the entire construction already presented for $\mathcal{L}(\mathcal{A})$, but we only emphasize the major differences.

We keep the notations introduced before with the only differences that for an arbitrary $\phi \in \mathcal{L}^+(\mathcal{A})$, the definition of the modal depth of ϕ also includes $md(M_r^a\psi) = md(\psi) + 1$ and $gr(\phi)$, $max(\phi)$ and $act(\phi)$ take into account, in addition, the indexes of the operators of type M_r^a that appear in ϕ. With these modifications we define $\mathcal{L}_n^+(A)$ for any integer n and $A \subseteq \mathcal{A}$ as before and for $\Lambda \subseteq \mathcal{L}^+(\mathcal{A})$, $[\Lambda]_n = \{\phi \in \mathcal{L}_n^+(\mathcal{A}) : \Lambda \vdash \phi\}$.

Now, we consider a consistent formula $\psi \in \mathcal{L}^+(\mathcal{A})$ with $gr(\psi) = n$ and $act(\psi) = A$ and we define $\mathcal{L}^+[\psi] = \{\phi \in \mathcal{L}_n^+(A) \mid max(\phi) \leq max(\psi), md(\phi) \leq md(\psi)\}$. For $\Omega[\psi]$ the set of $\mathcal{L}^+[\psi]$-maximally consistent sets of formulas we reproduce identically the construction done in the previous subsection for $\mathcal{L}(\mathcal{A})$.

The first important difference with respect to the previous case appears due to (B2): for each $\Lambda \in \Omega[\psi]$, $\phi \in \Lambda$ and $a \in \mathcal{A}$, there exist $s, t \in \mathbb{Q}_p$, $s < t$, such that $L_s^a\phi, M_r^a\phi \in \Lambda^+$. Secondly, for any $\Gamma \in \Omega_p[\psi]$, $\phi \in \mathcal{L}^+[\psi]$ and $a \in A$, there exist $x = max\{r \in \mathbb{Q}_p : \neg M_r^a\phi \in \Gamma\}$, $y = min\{r \in \mathbb{Q}_p : M_r^a\phi \in \Gamma\}$ and $y = x + 1/p$. In effect, in the correspondent of Lemma 7, one can prove that there exists $z = sup\{r \in \mathbb{Q} : L_r^a\phi \in \Gamma^\infty\} = inf\{r \in \mathbb{Q} : \neg L_r^a\phi \in \Gamma^\infty\} = inf\{r \in \mathbb{Q} : M_r^a\phi \in \Gamma^\infty\} = sup\{r \in \mathbb{Q} : \neg M_r^a\phi \in \Gamma^\infty\}$.

As before, we denote z by a_ϕ^Γ and we proceed with the definition of the model $\mathcal{M}_\psi$.

▶ **Lemma 14.** *If $\theta_\psi : \mathcal{A} \to [\Omega_p[\psi] \to \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]$ is defined for arbitrary $a \in \mathcal{A}$, $\Gamma \in \Omega_q[\psi]$ and $\phi \in \mathcal{L}^+[\psi]$ by $\theta_\psi(a)(\Gamma)(\llbracket\phi\rrbracket) = a_\phi^\Gamma$, then $\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi) \in \mathfrak{M}$.*

This last result allows us to prove the Truth Lemma for $\mathcal{L}^+(\mathcal{A})$.

▶ **Lemma 15** (Truth Lemma). *If $\phi \in \mathcal{L}^+[\psi]$, then $[\mathcal{M}_\psi, \Gamma \Vdash \phi$ iff $\phi \in \Gamma]$.*

With respect to the proof of Lemma 9, Lemma 15 requires the case $\phi = M_r^a\phi'$, which is proved symmetrically with the case $\phi = L_r^a\phi'$ in Lemma 9.

As before, the truth lemma implies the finite model property and the completeness theorem for $\mathcal{L}^+(\mathcal{A})$ and Markovian semantics.

▶ **Theorem 16** (Small model property). *For any $\mathcal{L}^+(\mathcal{A})$-consistent formula ϕ, there exists $\mathcal{M} \in \mathfrak{M}$ with finite support of cardinality bound by the structure of ϕ, and there exists $m \in sup(\mathcal{M})$ such that $\mathcal{M}, m \Vdash \phi$.*

▶ **Theorem 17** (Completeness). *The axiomatic system of $\mathcal{L}^+(\mathcal{A})$ is complete with respect to the Markovian semantics, i.e. if $\Vdash \psi$, then $\vdash \psi$.*

6 From bisimulation to the metric space of logical formulas

For the beginning, we state that the logical equivalences induced by $\mathcal{L}(\mathcal{A})$ and by $\mathcal{L}^+(\mathcal{A})$ on the class of CMPs coincide with stochastic bisimulation. The proofs follow closely the proof of the corresponding result for probabilistic systems and consists in showing that the negation free-fragment of $\mathcal{L}(\mathcal{A})$ characterizes stochastic bisimulation while the negation and M_r^a do not differentiate bisimilar processes. Being the similarity of the proofs with the probabilistic case we only sketch them in the appendix. For the detailed proof in the probabilistic case, the reader is referred to [7, 9, 18].

▶ **Theorem 18** (Logical characterization of stochastic bisimulation). *Let $\mathcal{M} = (M, \Sigma, \tau), \mathcal{M}' = (M', \Sigma', \tau') \in \mathfrak{M}$, $m \in M$ and $m' \in M'$. The following assertions are equivalent.*

1. $(\mathcal{M}, m) \sim (\mathcal{M}', m')$*;*
2. For any $\phi \in \mathcal{L}(\mathcal{A})$*,* $\mathcal{M}, m \Vdash \phi$ *iff* $\mathcal{M}', m' \Vdash \phi$*;*
3. For any $\phi \in \mathcal{L}^+(\mathcal{A})$*,* $\mathcal{M}, m \Vdash \phi$ *iff* $\mathcal{M}', m' \Vdash \phi$*.*

One of the main motivation for studying quantitative logics for probabilistic and stochastic processes was, since the first papers on this subject [16, 15], the characterization of stochastic/probabilistic bisimulation. In the context of Theorem 18, one can turn the bisimulation question into a series of model-checking problems. But the concept of stochastic/probabilistic bisimulation is a very strict concept: it only verifies whether two processes have identical behaviours. In applications we need instead to know whether two processes that may differ by only a small amount in real-valued parameters (rates or probabilities) are behaving in a similar way. To solve this problem a class of pseudometrics have been proposed in the literature [5, 18], to measure how similar two processes are in terms of stochastic/probabilistic behaviour.

Because these pseudometrics are quantitative extensions of bisimulation, they can be defined relying on the quantitative logics. Thus, for a class $\mathfrak{P}$ of stochastic or probabilistic processes and for a quantitative logic $\mathcal{L}$ that characterizes the bisimulation of processes, the satisfiability relation $\Vdash: \mathfrak{P} \times \mathcal{L} \to \{0,1\}$ can be extended to a function $d : \mathfrak{P} \times \mathcal{L} \to [0,1]$ which measures the *"degree of satisfiability"*, as shown in, e.g., [5, 18].

An example of such a metric for our case, where $\mathfrak{P}$ is the set of CMPs and $\mathcal{L} = \mathcal{L}^+(\mathcal{A})$ (or $\mathcal{L} = \mathcal{L}(\mathcal{A})$), is given by $d : \mathfrak{P} \times \mathcal{L} \to [0,1]$ defined below.

$d((\mathcal{M}, m), \top) = 0$,
$d((\mathcal{M}, m), \neg\phi) = 1 - d((\mathcal{M}, m), \phi)$,
$d((\mathcal{M}, m), \phi \wedge \psi) = max\{d((\mathcal{M}, m), \phi), d((\mathcal{M}, m), \psi)\}$,
$d((\mathcal{M}, m), L_r^a\phi) = \langle r, \theta(a)(m)([\![\phi]\!])\rangle$,
$d((\mathcal{M}, m), M_r^a\phi) = \langle \theta(a)(m)([\![\phi]\!]), r\rangle$,
where for arbitrary $a, b \in \mathbb{R}_+$, $\langle a, b\rangle = (a-b)/a$ if $a(a-b) > 0$ and $\langle a, b\rangle = 0$ else.
The following lemma shows that, indeed, d characterizes stochastic bisimulation.

▶ **Lemma 19.** *If* $(\mathcal{M}, m), (\mathcal{M}', m') \in \mathfrak{P}$*, then*

$$(\mathcal{M}, m) \sim (\mathcal{M}', m') \text{ iff } [\forall\phi \in \mathcal{L}, d((\mathcal{M}, m), \phi) = d((\mathcal{M}', m'), \phi)].$$

Proof. ($\Longrightarrow$) Induction on ϕ. The Boolean cases are trivial and the cases $\phi = L_r^a\psi$ and $\phi = M_r^a\psi$ derive from the fact that $\theta(a)(m)([\![\phi]\!]) = \theta'(a)(m')([\![\phi]\!])$.
($\Longleftarrow$) For an arbitrary $\phi \in \mathcal{L}$, $\forall r \in \mathbb{Q}$, $d((\mathcal{M}, m), L_r^a\phi) = d((\mathcal{M}', m'), L_r^a\phi)$; and for r big enough $d((\mathcal{M}, m), L_r^a\phi) = 1 - \theta(a)(m)([\![\phi]\!])/r$, $d((\mathcal{M}', m'), L_r^a\phi) = 1 - \theta'(a)(m')([\![\phi]\!])/r$. Hence, $\theta(a)(m)([\![\phi]\!]) = \theta'(a)(m')([\![\phi]\!])$ which implies $(\mathcal{M}, m) \sim (\mathcal{M}', m')$. ◀

There exist a few such metrics defined in literature [5, 18], mainly for probabilistic systems, that take into account various intuitions about how one can quantify the satisfiability relation. However, any such function $d : \mathfrak{P} \times \mathcal{L} \to [0,1]$, if it characterizes bisimulation in the sense of Lemma 19, induces a distance between two stochastic processes, $D : \mathfrak{P} \times \mathfrak{P} \to [0,1]$ by

$$D(P, P') = sup\{|d(P,\phi) - d(P',\phi)|, \phi \in \mathcal{L}\}, \text{ for arbitrary } P, P' \in \mathfrak{P}.$$

An immediate consequence of Lemma 19 (or of a similar result) is the next lemma.

▶ **Lemma 20.** $D : \mathfrak{P} \times \mathfrak{P} \to [0,1]$ *defined before is a pseudometric such that*

$$D(P, P') = 0 \text{ iff } P \sim P'.$$

Similarly, one can use d to define a pseudometric $\overline{d} : \mathcal{L} \times \mathcal{L} \to [0,1]$ over the space of logical formulas by

$$\overline{d}(\phi, \psi) = sup\{|d(P, \phi) - d(P, \psi)|, P \in \mathfrak{P}\}, \text{ for arbitrary } \phi, \psi \in \mathcal{L}.$$

▶ **Lemma 21.** *$\overline{d} : \mathcal{L} \times \mathcal{L} \to [0,1]$ defined before is a pseudometric and*

$$\overline{d}(\phi, \psi) = \overline{d}(\neg\phi, \neg\psi).$$

The utility of $\overline{d}$ is stated by the robustness theorem.

▶ **Theorem 22** (Strong Robustness). *For arbitrary $\phi, \psi \in \mathcal{L}$ and $P \in \mathfrak{P}$,*

$$d(P, \psi) \leq d(P, \phi) + \overline{d}(\phi, \psi).$$

Proof. The inequality is equivalent to $d(P, \psi) - d(P, \phi) \leq \overline{d}(\phi, \psi)$, which derives from the definition of $\overline{d}$. ◀

Similar constructions can be done for any class of stochastic or probabilistic models for which it has been defined a correspondent logic that characterizes bisimulation. But in spite of the obvious utility of the robustness theorem, in most of the cases such a result is not computable due to the definition of $\overline{d}$ that involves the quantification over the entire class of continuous Markov processes.

This is exactly where the sound-complete axiomatizations of $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^+(\mathcal{A})$ for the Markovian semantics and the finite model properties play their role. In what follows, we use the construction of the small model for an $\mathcal{L}$-consistent formula presented in the previous section[5] to effectively compute an approximation of $\overline{d}$ within a given error $\varepsilon > 0$. Bellow we reuse the notations used in section 5.

Let Ω be the set of the $\mathcal{L}$-maximally consistent sets of formulas. For arbitrary $\Gamma^\infty \in \Omega$, $a \in \mathcal{A}$ and $\phi \in \mathcal{L}$, let

$$a_\phi^{\Gamma^\infty} = sup\{r \in \mathbb{Q} : L_r^a \phi \in \Gamma^\infty\} = inf\{r \in \mathbb{Q} : \neg L_r^a \phi \in \Gamma^\infty\} =$$
$$inf\{r \in \mathbb{Q} : M_r^a \phi \in \Gamma^\infty\} = sup\{r \in \mathbb{Q} : \neg M_r^a \phi \in \Gamma^\infty\}.$$

The existence of these inf and sup and their equalities can be proved as in Lemma 6 (4).

▶ **Lemma 23** (Extended Truth Lemma). *If $\theta : \mathcal{A} \to [\Omega \to \Delta(\Omega, 2^\Omega)]$ is defined for arbitrary $a \in \mathcal{A}$, $\Gamma^\infty \in \Omega$ and $\phi \in \mathcal{L}$ by $\theta(a)(\Gamma^\infty)([\![\phi]\!]) = a_\phi^{\Gamma^\infty}$, then $\mathcal{M}_\mathcal{L} = (\Omega, 2^\Omega, \theta) \in \mathfrak{M}$. Moreover, for arbitrary $\phi \in \mathcal{L}$,*

$$\mathcal{M}_\mathcal{L}, \Gamma^\infty \Vdash \phi \text{ iff } \phi \in \Gamma^\infty.$$

The proof of this lemma is the sum of the proofs of the lemmas 8, 9, 14 and 15.

The next lemma states that $\overline{d}$ can be characterized by only using the processes of $\mathcal{M}_\mathcal{L}$. Because these processes are $\mathcal{L}$-maximally consistent sets of formulas, the next lemma is basically showing that $\overline{d}$ is a distance depending directly on provability. From the computability point of view, the reduction of the space of quantification is not simplifying our problem as Ω is itself infinite.

▶ **Lemma 24.** *For arbitrary $\phi, \psi \in \mathcal{L}$,*

$$\overline{d}(\phi, \psi) = sup\{|d((\mathcal{M}_\mathcal{L}, \Gamma^\infty), \phi) - d((\mathcal{M}_\mathcal{L}, \Gamma^\infty), \psi)|, \Gamma^\infty \in \Omega\}.$$

[5] The results presented bellow are true for both $\mathcal{L} = \mathcal{L}(\mathcal{A})$ and $\mathcal{L} = \mathcal{L}^+(\mathcal{A})$.

Now we try to reduce the quantification space even more, to the domain of a finite model. For an arbitrary consistent formula $\psi \in \mathcal{L}$, let $\mathcal{M}_\psi = (\Omega_p[\psi], 2^{\Omega_p[\psi]}, \theta_\psi) \in \mathfrak{M}$ be the model of ψ constructed in the previous section; we call p the *parameter of* $\mathcal{M}_\psi$.
Let $\widetilde{d} : \mathcal{L} \times \mathcal{L} \to [0, 1]$ be defined as follows.
$\widetilde{d}(\phi, \psi) = max\{|d((\mathcal{M}_{\phi\wedge\psi}, \Gamma), \phi) - d((\mathcal{M}_{\phi\wedge\psi}, \Gamma), \psi)|, \Gamma \in \Omega_p[\phi \wedge \psi]\}$ if $\phi \wedge \psi$ is consistent,
$\widetilde{d}(\phi, \psi) = max\{|d((\mathcal{M}_{\neg(\phi\wedge\psi)}, \Gamma), \phi) - d((\mathcal{M}_{\neg(\phi\wedge\psi)}, \Gamma), \psi)|, \Gamma \in \Omega_p[\neg(\phi \wedge \psi)]\}$ else.

▶ **Lemma 25.** *For arbitrary $\phi, \psi \in \mathcal{L}$,*

$$\overline{d}(\phi, \psi) \leq \widetilde{d}(\phi, \psi) + 2/p.$$

This last result finally allows us to prove a weaker version of the robustness theorem which evaluates $d((\mathcal{M}, m), \psi)$ from $d((\mathcal{M}, m), \phi)$, based on $\widetilde{d}(\phi, \psi)$ and a given error.

▶ **Theorem 26** (Weak Robustness). *For arbitrary $\phi, \psi \in \mathcal{L}$ and $P \in \mathfrak{P}$,*

$$d(P, \psi) \leq d(P, \phi) + \widetilde{d}(\phi, \psi) + 1/p,$$

where p is the parameter of $\mathcal{M}_{\phi\wedge\psi}$ if $\phi \wedge \psi$ is consistent, or of $\mathcal{M}_{\neg(\phi\wedge\psi)}$ otherwise.

Because $\mathcal{M}_{\phi\wedge\psi}$ (or $\mathcal{M}_{\neg(\phi\wedge\psi)}$) is finite, $\widetilde{d}(\phi, \psi)$ can be computed and the error $1/p$ can also be controlled while constructing $\mathcal{M}_{\phi\wedge\psi}$. Hence, we can evaluate $d(P, \phi)$ from $d(P, \psi)$. This is obviously useful when P is infinite or very large and it is expensive to repeatedly evaluate $d(P, \phi)$ for various ϕ. Instead, our theorem allows us to evaluate $d(P, \psi)$ from $d(P, \phi)$ that we can get, for instance, using statistical model checking techniques.

7 Conclusions and future works

In this paper we introduce Continuous Markovian Logic, a multimodal logic designed to specify quantitative and qualitative properties of Markov processes with continuous state-space and continuous-time transitions. CML is endowed with operators that approximate the rates of the labelled transition of CMPs and allows us to approximate properties. This logic, as in the probabilistic case, characterizes the stochastic bisimulation of CMPs. We present two sound-complete Hilbert-style axiomatizations: for CMP and for CMP without M^a_r-operators. These axiomatic systems are significantly different from the probabilistic case and from each other. The two completeness proofs presented in the paper relay on the finite model properties that we prove for both CML and its restricted fragment. The constructions of the finite models uses the filtration method of modal logics in the stochastic settings, where a series of original problems have been solved. The small model construction and the complete axiomatization allows us to approach the problems of bisimulation-distances, that in the probabilistic cases were only approached semantically, from a syntactic perspective. In effect we can define a distance between logical formulas that allows us to prove the robustness theorems.

This paper opens a series of interesting research questions regarding the relation between satisfiability, provability and metric semantics. There are many open questions related to the definition of $\overline{d}$ and the structure of the metric space of formulas. One of the problems, that we postpone for future work, is finding a classification of the functions d to reflect properties of $\overline{d}$. For instance, we have a partial result showing that if d is such that $[d(P, \phi) = 0$ iff $P \Vdash \phi]$[6], then $\overline{d}$ characterizes the logical equivalence, i.e., $[\overline{d}(\phi, \psi) = 0$ iff $\Vdash \phi \leftrightarrow \psi]$. There

[6] In general, if the distance d satisfies a rule of type $d(P, \neg\phi) = 1 - d(P, \phi)$, as the metric which we use as an example in this paper does, it does not enjoy this property.

exist, however, distances enjoying even stronger properties such as [$\Vdash \phi \to \psi$ iff $\forall P \in \mathfrak{P}$, $d(P,\psi) \leq d(P,\phi)$]. Each of these metrics organizes the set of logical formulas as a metric space, in the way we have shown in our paper, and in each case the metric space has different properties. The complete axiomatization is probably the key for understanding the relationship between these structures.

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Appendix

In this appendix we have collected the proofs of the major results and the detailed discussions of the examples presented in the paper.

Lemma 6. The existence of x and y derives from the construction of $\Omega_p[\psi]$ and the Rules (R2), (R3).
Because Γ is consistent and $L_x^\alpha\phi, \neg L_y^\alpha\phi \in \Gamma$, $x \neq y$. If $x > y$, $L_x^\alpha\phi \in \Gamma$ entails (Axiom (A2)) $L_y^\alpha\phi \in \Gamma$, contradicting the consistency of Γ.
Hence, $x < y$. If $x + 1/p < y$, then $L_{x+1/p}^\alpha\phi \notin \Gamma$ (because $x < x + 1/p \in \mathbb{Q}_q$ and Γ is $\mathcal{L}_p$-maximally consistent), i.e. $\neg L_{x+1/p}^\alpha\phi \in \Gamma$ implying that $x + 1/p \geq y$ - contradiction. ◀

Lemma 7. As before, the existence of sup and inf is guaranteed by the construction and the Rules (R2) and (R3).

Let $x^\infty = sup\{r \in \mathbb{Q} : L_r^a\phi \in \Gamma^\infty\}$ and $y^\infty = inf\{r \in \mathbb{Q} : \neg L_r^a\phi \in \Gamma^\infty\}$. Suppose that $x^\infty < y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty < r < y^\infty$. This implies that $\neg L_r^a\phi \in \Gamma^\infty$ (from the definition of x^∞) and $L_r^a\phi \in \Gamma^\infty$ (from the definition of y^∞) - impossible because Γ^∞ is consistent.

Suppose that $x^\infty > y^\infty$. Then there exists $r \in \mathbb{Q}$ such that $x^\infty > r > y^\infty$. As Γ^∞ is maximally consistent we have either $L_r^a\phi \in \Gamma^\infty$ or $\neg L_r^a\phi \in \Gamma^\infty$. The first case contradicts the definition of x^∞ while the second the definition of y^∞.

Obviously, $x \leq z \leq y$. We cannot have $z = y$ because else $L_z^a\phi, \neg L_z^a\phi \in \Gamma$ contradicting the consistency of Γ. ◀

Lemma 8. This result is a direct consequence of the construction of $\mathcal{M}_\psi$. First notice that because the space is discrete, is Polish, hence analytic set.

The central problem is to prove that for arbitrary $\Gamma \in \Omega_p[\psi]$ and $a \in A$, the function $\theta_\psi(a)(\Gamma) : 2^{\Omega_p[\psi]} \to \mathbb{R}^+$ is well defined and a measure on $(\Omega_p[\psi], 2^{\Omega_p[\psi]})$. Further, because the space is discrete with finite support, we obtain that $\theta_\psi(a) \in [\![\Omega_p[\psi] \to \Delta(\Omega_p[\psi], 2^{\Omega_p[\psi]})]\!]$ and conclude the proof.

$\theta_\psi(a)(\Gamma)$ is well defined: suppose that for $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ we have $[\![\phi_1]\!] = [\![\phi_2]\!]$. Then, from Lemma 6, $\vdash \phi_1 \leftrightarrow \phi_2$ and from Rule (R1) $\vdash L_r^a\phi_1 \leftrightarrow L_r^a\phi_2$. This entails $a_{\phi_1}^\Gamma = a_{\phi_2}^\Gamma$ and guarantees that $\theta_\psi(a)(\Gamma)$ is well defined.

Now we prove that $\theta_\psi(a)(\Gamma)$ is a measure.

For showing $\theta_\psi(a)(\Gamma)(\emptyset) = 0$, we show that for any $r > 0$, $\vdash \neg L_r^a\bot$. This is sufficient, as Axiom (A1) guarantees that $\vdash L_0^a\bot$ and $[\![\bot]\!] = \emptyset$. Suppose that there exists $r > 0$ such that $L_r^a\bot$ is consistent. Let $\epsilon \in (0, r) \cap \mathbb{Q}$. Then Axiom (A2) gives $\vdash L_r^a\bot \to L_\epsilon^a\bot$. Hence, $\vdash L_r^a\bot \to (L_r^a(\bot \wedge \bot) \wedge L_\epsilon^a(\bot \wedge \neg\bot))$ and applying the Axiom (A3), $\vdash L_r^a\bot \to L_{r+\epsilon}^a\bot$. Repeating this argument, we can prove that $\vdash L_r^a\bot \to L_s^a\bot$ for any s and Rule (R3) confirms the inconsistency of $L_r^a\bot$.

We show now that if $A, B \in 2^{\Omega_p[\psi]}$ with $A \cap B = \emptyset$, then $\theta_\psi(a)(\Gamma)(A) + \theta_\psi(a)(\Gamma)(B) = \theta_\psi(a)(\Gamma)(A \cup B)$. Let $A = [\![\phi_1]\!]$, $B = [\![\phi_2]\!]$ with $\phi_1, \phi_2 \in \mathcal{L}[\psi]$ and $\vdash \phi_1 \to \neg\phi_2$. Let $x_1 = \theta_\psi(a)(\Gamma)(A)$, $x_2 = \theta_\psi(a)(\Gamma)(B)$ and $x = \theta_\psi(a)(\Gamma)(A \cup B)$. We prove that $x_1 + x_2 = x$.

Suppose that $x_1 + x_2 < x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x_1' + x_2' < x$, where $x_i' = x_i + \epsilon_i$ for $i = 1, 2$. But this implies that $L_{x_i'}^a\phi_i \notin \Gamma^\infty$ (from the definition of x_i), hence

$\neg L^a_{x'_i}\phi_i \in \Gamma^\infty$. Further, using Axiom (A4), we obtain $\neg L^a_{x'_1+x'_2}(\phi_1 \vee \phi_2) \in \Gamma^\infty$, implying (from the definition of x) that $x'_1 + x'_2 \geq x$ - contradiction.

Suppose that $x_1 + x_2 > x$. Then, there exist $\epsilon_1, \epsilon_2 \in \mathbb{Q}^+$ such that $x''_1 + x''_2 > x$, where $x''_i = x_i - \epsilon_i$ for $i = 1, 2$. But this implies (from the definition of x_i) that $L^a_{x''_i}\phi_i \in \Gamma^\infty$. Further, Axiom (A3) gives $L^a_{x''_1+x''_2}(\phi_1 \vee \phi_2) \in \Gamma^\infty$, i.e. $x''_1 + x''_2 \leq x$ - contradiction.

◀

Theorem 18. The fact that bisimilar processes satisfy the same formulas can be proved by a trivial induction on the structure of the logical formula.

Further we prove that the logical equivalence induced by the negation-free fragment $\mathcal{L}^*(\mathcal{A})$ of $\mathcal{L}(\mathcal{A})$ characterizes completely the stochastic bisimulation of CMPs. For simplicity, consider for arbitrary $\mathcal{A}$-CMPs $\mathcal{M} = (M, \Sigma, \tau)$, $\mathcal{M}' = (M', \Sigma', \tau')$ and $m \in M$, $m' \in M'$ the relation $\approx$ defined by $(\mathcal{M}, m) \approx (\mathcal{M}', m')$ iff [for any $\phi \in \mathcal{L}^*(\mathcal{A})$, $\mathcal{M}, m \Vdash \phi$ iff $\mathcal{M}', m' \Vdash \phi$]. We will show that if $(\mathcal{M}, m) \approx (\mathcal{M}', m')$, then $(M, m) \sim (\mathcal{M}', m')$. For this, we show that $\approx$ is a rate-bisimulation.

Before starting this proof, we introduce some additional concepts and present some results that are needed for our proof.

Given a set X, a family of subsets $\Pi \subset 2^X$ closed under finite intersection is called *π-system*. A family of subsets $\Lambda \subset 2^X$ is a *λ-system* if contains X and is closed under complementation and countable union of pairwise disjoint sets.

[Dynkin's $\lambda - \pi$ theorem]. *If Π is a π-system and Λ is a λ-system, then $\Pi \subset \Lambda$ implies $\overline{\Pi} \subseteq \Lambda$, where $\overline{\Pi}$ is the σ-algebra generated by Π.*

This theorem allows us to prove the next lemma.

[Lemma A.] *Suppose that $\Pi \subseteq 2^X$ is a π-system with $X \in \Pi$ and μ, ν are two measures on $(X, \overline{\Pi})$. If μ and ν agree on all the sets in Π, then they agree on $\overline{\Pi}$.*

We also present two more lemmas (see, e.g., [18] Section 7.7).

[Lemma B.] *Let (M, Σ) be an analytic space and let Σ_0 be a countably generated sub-σ-algebra of Σ which separates points in M, i.e., for any $m, n \in M$, $m \neq n$, there exists $S \in \Sigma_0$ such that $m \in S \not\ni n$. Then $\Sigma_0 = \Sigma$.*

[Lemma C.] *Let (M, Σ) be an analytic space and let $\equiv$ be an equivalence relation on M. If there exists a sequence $f_1, f_2, \ldots$ of real-valued Borel functions on M such that $m \equiv n$ iff for all i, $f_i(m) = f_i(n)$, then $(M^\equiv, \Sigma^\equiv)$ is an analytic space.*

Now we are ready to prove that $\approx$ is a rate-bisimulation. We introduce the concept of *zigzag morphism* for CMPs, similar to [7, 18], which is a functional analogue of the concept of bisimulation and will be the cornerstone of the completeness proof.

▶ **Definition 27** (Zigzag morphism)**.** A function f from $\mathcal{M} = (M, \Sigma, \theta)$ to $\mathcal{M}' = (M', \Sigma', \theta')$ is a zigzag morphism if it is surjective, measurable and for all $\alpha \in \mathcal{A}$, $m \in M$ and $S' \in \Sigma'$,

$$\theta(\alpha)(m)(f^{-1}(S')) = \theta'(\alpha)(f(m))(S').$$

Notice that $\approx$ is an equivalence relation, hence, for a given (M, Σ) we can consider the quotient $(M^\approx, \Sigma^\approx)$ constructed as follows. $M^\approx$ is the set of all equivalence classes of M; there exists a projection $\pi : M \to M^\approx$ which maps each element to its equivalence class. π determines a σ-algebra $\Sigma^\approx$ on $M^\approx$ by $S \in \Sigma^\approx$ iff $\pi^{-1}(S) \in \Sigma$. We call π the *canonical projection* from (M, Σ) into $(M^\approx, \Sigma^\approx)$.

For the beginning we show that $(M^\approx, \Sigma^\approx)$ is an analytic space. Let $\mathcal{L}^*(\mathcal{A}) = \{\phi_i | i \in \mathbb{N}\}$. Because $[\![\phi_i]\!]_\mathcal{M}$ is measurable, the characteristic functions $1_{\phi_i} : M \to \{0, 1\}$ are measurable and $m \approx n$ iff $[\forall i \in \mathbb{N}, 1_{\phi_i}(m) = 1_{\phi_i}(n)]$. Lemma C proves further that $(M^{\approx}, \Sigma^{\approx})$ is an analytic space.

Let $\mathcal{B} = \{\pi([\![\phi_i]\!]_{\mathcal{M}} | i \in \mathbb{N}\}$. We show that $\overline{\mathcal{B}} = \Sigma^{\approx}$. Obviously, $\mathcal{B} \subseteq \Sigma^{\approx}$, because for any $\pi([\![\phi_i]\!]_{\mathcal{M}}) \in \mathcal{B}$, $\pi^{-1}(\pi([\![\phi_i]\!]_{\mathcal{M}})) \in \Sigma$. Notice that $\overline{\mathcal{B}}$ separates points in $M^{\approx}$: let $C, D \in M^{\approx}$, $C \neq D$ and let $m \in \pi^{-1}(C)$, $n \in \pi^{-1}(D)$; because $m \not\approx n$, there exists $\phi \in \mathcal{L}^*(\mathcal{A})$ such that $m \in [\![\phi]\!]_{\mathcal{M}} \not\ni n$. Hence, we can apply Lemma B and we obtain $\overline{\mathcal{B}} = \Sigma^{\approx}$.

Now we define $\theta^{\approx}$ such that π is a zigzag. Notice first that π is measurable and surjective by definition. For each $C \in \Sigma^{\approx}$ and $\alpha \in \mathcal{A}$, let $\theta^{\approx}(\alpha)(m^{\approx})(C) = \theta(\alpha)(m)(\pi^{-1}(C))$.

This definition is correct: let $m, n \in m^{\approx}$, we prove that $\theta(\alpha)(m)$ and $\theta(\alpha)(n)$ agree on $\Sigma^{\approx}$. We show first that they agree on $[\![\phi]\!]_{\mathcal{M}} \in \mathcal{B}$. Suppose that we have $\theta(\alpha)(m)([\![\phi]\!]_{\mathcal{M}}) < r < \theta(\alpha)([\![\phi]\!]_{\mathcal{M}})$. Then, $\mathcal{M}, m \Vdash \neg L_r^{\alpha}\phi$ while $\mathcal{M}, n \Vdash L_r^{\alpha}\phi$ - impossible. Because $\mathcal{B}$ is closed under finite intersection ($[\![\phi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}} = [\![\phi \wedge \psi]\!]_{\mathcal{M}}$) and $M = [\![\top]\!]_{\mathcal{M}} \in \mathcal{B}$, we apply Lemma A and obtain that $\theta(\alpha)(m)$ and $\theta(\alpha)(n)$ agree on $\Sigma^{\approx}$.

Now we only need to prove that for any $\alpha \in \mathcal{A}$, $\theta^{\approx}(\alpha)$ is measurable. Let $C \in \Sigma^{\approx}$ and A a Borel set of $\mathbb{R}^+$. We have

$$(\theta^{\approx})^{-1}(\{\mu \in \Delta(M^{\approx}, \Sigma^{\approx}) | \mu(B) \in A\}) = \pi((\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\})).$$

But $\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\}$ is measurable in $\Delta(M, \Sigma)$ and because $\theta(\alpha)$ is measurable we obtain that $(\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\}) \in \Sigma$ implying $\pi((\theta(\alpha))^{-1}(\{\nu \in \Delta(M, \Sigma) | \nu(\pi^{-1}(B)) \in A\})) \in \Sigma^{\approx}$. ◀

Lemma 21. It is sufficient to show that it satisfies the triangle inequality. We have $sup\{|d((\overline{\Omega}, \Gamma), \phi) - d((\overline{\Omega}, \Gamma), \psi)|\} + sup\{|d((\overline{\Omega}, \Gamma), \psi) - d((\overline{\Omega}, \Gamma), \rho)|\} \geq sup\{|d((\overline{\Omega}, \Gamma), \phi) - d((\overline{\Omega}, \Gamma), \psi)| + |d((\overline{\Omega}, \Gamma), \psi) - d((\overline{\Omega}, \Gamma), \rho)|\} \geq sup\{|d((\overline{\Omega}, \Gamma), \phi) - d((\overline{\Omega}, \Gamma), \rho)|\}$. ◀

Lemma 24. Any $(\mathcal{M}, m) \in \mathfrak{M}$ satisfies a maximally-consistent set of formulas, hence there exists $\Gamma^{\infty} \in \Omega$ such that $(\mathcal{M}, m) \sim (\mathcal{M}_{\mathcal{L}}, \Gamma^{\infty})$, i.e., for any $\phi \in \mathcal{L}$, $d((\mathcal{M}, m), \phi) = d((\mathcal{M}_{\mathcal{L}}, \Gamma^{\infty}), \phi)$. ◀

Lemma 25. To prove the inequality, we return to the notations of lemmas 6 and 7. We have $x, y \in \mathbb{Q}_p$, $y = x + 1/p$ and $x \leq z < y$. This implies that for any $\phi \in \mathcal{L}[\psi]$, $|d((\mathcal{M}_{\mathcal{L}}, \Gamma^{\infty}), \phi) - d((\mathcal{M}_{\psi}, \Gamma), \phi)| \leq 1/p$. Consequently, for arbitrary $\phi, \psi \in \mathcal{L}$, $|d((\mathcal{M}_{\mathcal{L}}, \Gamma^{\infty}), \phi) - d((\mathcal{M}_{\mathcal{L}}, \Gamma^{\infty}), \psi)| \leq |d((\mathcal{M}_{\phi\wedge\psi}, \Gamma), \phi) - d((\mathcal{M}_{\phi\wedge\psi}, \Gamma), \psi)| + 2/p$, which proofs our inequality. ◀