Modular Markovian Logic

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Motivation

Complex networks/systems are often modelled as stochastic processes

- to encapsulate a lack of knowledge or inherent non-determinism,
- to model hybrid real-time and discrete-time interacting components,
- to abstract complex continuous-time and continuous-space systems.

Such systems are frequently modular in nature

- consist of parts which are systems in their own right,
- their global behaviour depends on the behaviour of their parts and on the links which connect them,
- the modules are easier to test/measure/analyse,
- often the knowledge of some modules is the only information available.

Such systems are extremely complex and large

- the classic verification techniques, designed to analyse complete systems, are often inefficient for study and predict their behaviours.
- Instead, various "*ad hoc sematic tricks*" are used to handle modularity.

Motivation

The role of modularity

- local knowledge sometimes provides global information

e.g., the presence of a *promoter*, the absence of a *catalyst* or the occurrence of a *triggering event* entails certain global behaviours

- concurrent behaviours are sometimes essential for explaining patterns of behaviour

e.g., often oscillating behaviours can be explained by synchronised actions of two modules with antagonist non-oscillatory behaviours

- modularity can prove properties in systems with unknown or inaccessible parts

e.g., communication and security protocols on complex networks, predictions for natural systems where the information is always incomplete

- modularity can be used to test causality scenarios

e.g., one can test the degree of connectivity between modules and predict the existence of various elements in inaccessible parts of the system

To what extent is it possible to predict the behaviour of a complex system or to prove some of its properties from the local observations of its modules?

$$\frac{P_1 \models f_1,\ P_2 \models f_2,\ \ldots,\ P_k \models f_k}{P_1|P_2|\ldots|P_k \models f} \quad C(f,f_1,f_2,\ldots,f_k)$$

A General Pattern for a Theory of Systems

The behaviour

- Systems are reactive – (unlike algorithms) they do not terminate their evolution and announce a result; they run "forever" and communicate, while running with their environment.
- Systems have interfaces – used for describing the possible communication with the environment; an external observer can observe a system only through the interface.
- The black box view of a system – the external observer's view. The black box view is given by the overall observable behaviour of the system.

The structure

- The systemic view comprises the concept of modularity/compositionality:
- A system is a network of modules

(subsystems – independent units of behaviour/computation).

- The modules communicate, interact or interrupt each other.
- The behaviour of a system emerges from the behaviours of its subsystems.

A General Pattern for a Theory of Systems

The behavior of a system is defined by a $\mathfrak{B}$-coalgebra, for an endofunctor $\mathfrak{B}$.

$$M \xrightarrow{\theta} \mathfrak{B}M$$

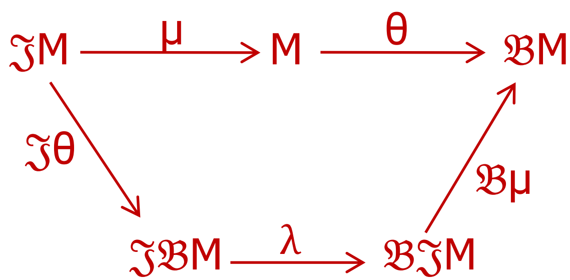
The structure of a system is defined by a $\mathfrak{I}$-algebra for an endofunctor $\mathfrak{I}$.

$$\mathfrak{I}M \xrightarrow{\mu} M$$

Compositionality: λ (a natural transformation between $\mathfrak{I}$ and $\mathfrak{B}$) defines a GSOS

$\Longrightarrow$ **$\mathfrak{I}\mathfrak{B}$-Bialgebra**

D. Turi, G. Plotkin, *Towards a mathematical operational semantics*, LICS'97



A logic for bialgebraic semantics should express both algebraic & coalgebraic properties.

The structure of the talk

- We introduce a general concept of **modular continuous Markov process** (MMP) that extends Panangaden's **continuous Markov process** to a Bialgebraic structure.

  parallel composition => **commutative monoid**.

- **stochastic bisimulation** => **structural bisimulation** (sensitive to modularity).

- We introduce the **Modular Markovian Logic** (MML) for MMPs

  MML expresses both stochastic and modular properties of systems.

- For MML, **logical equivalence=structural bisimulation**

- We present a **Hilbert-style axiomatization** for MML that is sound-complete w.r.t. Markovian semantics

- We prove some **metaproperties** of MML.

Focusing on behaviors… coalgebraically

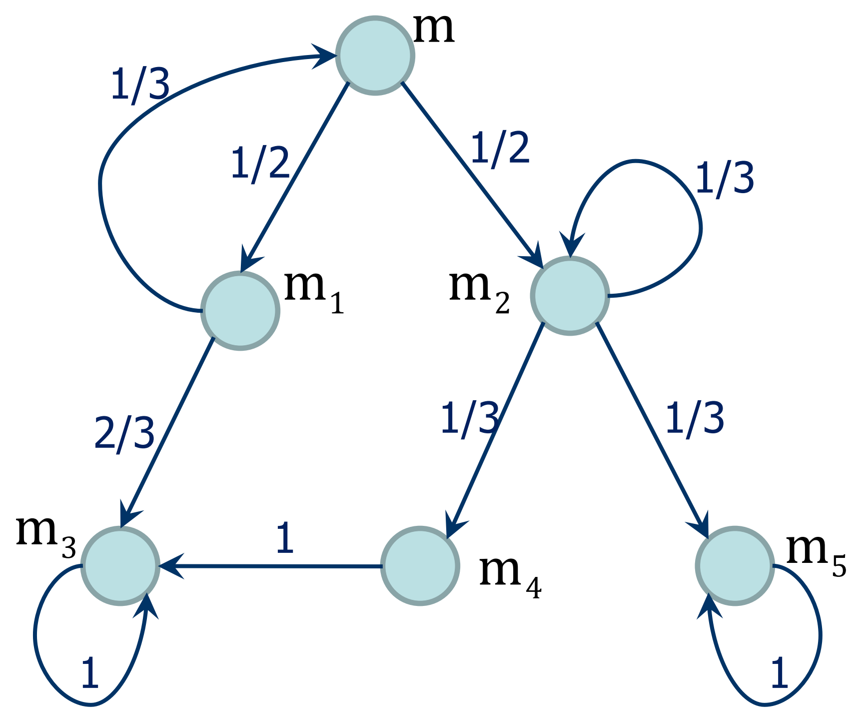
Markov chain

a tuple $\mathcal{M}$=(M,R) where

- M is a (countable) set of states
- R:M×M→[0,1] - probability matrix

  for each m∈M, $\sum_{m'\in M} R(m,m')=1$

If R(m,m')=p∈[0,1], we write $m \xrightarrow{p} m'$



Equivalent representation:

m ↦ μ, μ: 2^M→[0,1] – probabilistic distribution

e.g.,
μ({m,m_1})=1/2,
μ({m,m_1,m_3})=1/2,
μ({m_3,m_4})=0

Markov chain

$\mathcal{M}$=(M,2^M,θ), θ∈[M → Π(M,2^M)],

θ(m): 2^M→[0,1]

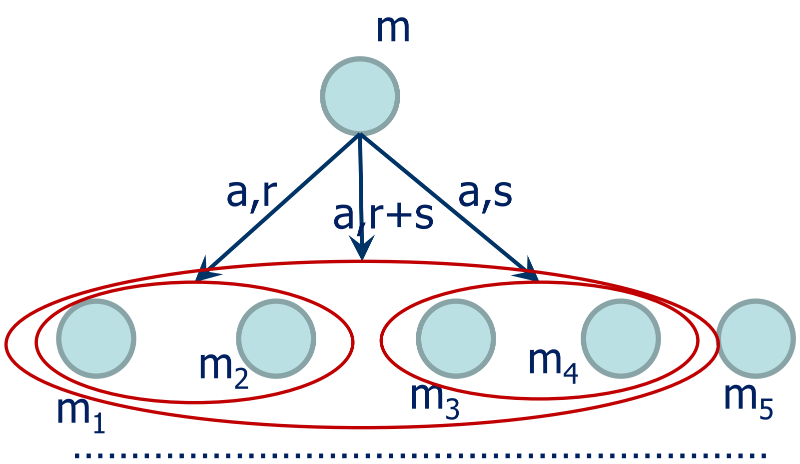
Focusing on behaviors… coalgebraically

Labelled Markov kernel

a tuple $\mathcal{M}=(M,\Sigma,A,\{R_a|a\in A\})$ where

- (M,Σ) is an analytic set (measurable space)
- Σ is the Borel-algebra generated by the topology
- A is a set of labels
- for each $a\in A$, $R_a:M\times\Sigma\to[0,1]$ is such that
  - $R_a(m,-)$ - (sub-)probability measure on (M,Σ)
  - $R_a(-,S)$ - measurable function

(P. Panangaden, *Labelled Markov Processes*, 2009.)



$m \xrightarrow{a,r} \{m_1,m_2\}$, $m \xrightarrow{a,s} \{m_3,m_4\}$

$m \xrightarrow{a,r+s} \{m_1,m_2,m_3,m_4\}$

$m \not\rightarrow \{m_2,m_3\}$

<u>Equivalent representation:</u>

$(m,a) \mapsto \mu_a$, $\mu_a: \Sigma\to[0,1]$, $\mu_a(S)=R_a(m,S)$

$\mu_a(\{m_1,m_2\})=r$, $\mu_a(\{m_3,m_4\})=s$,

$\mu_a(\{m_1,m_2,m_3,m_4\})=r+s$

Labelled Markov kernel

$\mathcal{M}=(M,\Sigma,\theta)$, $\theta\in[\![M \to \Pi(M,\Sigma)]\!]^A$

(E. Doberkat, *Stochastic Relations*, 2007.)

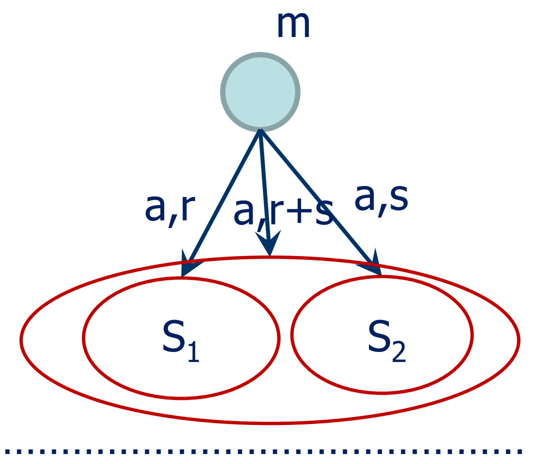
Focusing on behaviors… coalgebraically

Continuous (Labelled) Markov kernel

a tuple $\mathcal{M}=(M,\Sigma,A,\{R_a|a\in A\})$ where

- (M,Σ) is an analytic set (measurable space)
- A is a set of labels
- for each $a\in A$, $R_a:M\times\Sigma \to [0,\infty)$ is such that
    - $R_a(m,-)$ – a measure on (M,Σ)
    - $R_a(-,S)$ – a measurable function

J. Desharnais, P. Panangaden, *Continous Stochastic Logic Characterizes Bisimulation of Continuous-time Markov Processes*, 2003.



$m \xrightarrow{a,r} S_1$, $m \xrightarrow{a,s} S_2$, $m \xrightarrow{a,r+s} S_1 \cup S_2$

- $R_a(m,S)=r \in [0,+\infty)$ - the rate of an exponentially distributed random variable that characterizes the time of a-transitions from m to arbitrary elements of S.
- the probability of the *transition within time t* is given by the cumulative distribution function

$$P(t)= 1- e^{-rt}$$

Equivalent representation:

Continuous (Labelled) Markov kernel $\mathcal{M}=(M,\Sigma,\theta)$, $\theta\in[\![M \to \Delta(M,\Sigma)]\!]^A$

$\theta_a: M \to \Delta(M,\Sigma)$, $\theta_a(m)\in\Delta(M,\Sigma)$, $\theta_a(m)(S)\in [0,+\infty)$

Continuous Markov process $(\mathcal{M},m)$, $m\in M$

Focusing on behaviors... coalgebraically

Unlabeled transition systems

$\mathcal{M}=(M,2^M,\theta)$, $\theta \in [M \to D(M,2^M)]$

Labeled transition systems

$\mathcal{M}=(M,2^M,\theta)$, $\theta \in [M \to D(M,2^M)]^A$

Markov chain

$\mathcal{M}=(M,2^M,\theta)$, $\theta \in [M \to \Pi(M,2^M)]$

Reactive probabilistic automata

$\mathcal{M}=(M, 2^M,\theta)$, $\theta \in [M \to \Pi(M,2^M)]^A$

Labeled Markov kernel

$\mathcal{M}=(M,\Sigma,\theta)$, $\theta \in [\![M \to \Pi(M,\Sigma)]\!]^A$

Discrete stochastic transition systems

$\mathcal{M}=(M,2^M,\theta)$, $\theta \in [\![M \to \Delta(M,2^M)]\!]^A$

Continuous Markov kernel

$\mathcal{M}=(M,\Sigma,\theta)$, $\theta \in [\![M \to \Delta(M,\Sigma)]\!]^A$

(J. M. M. Rutten, Universal coalgebra: a theory of systems, 2000.)

The behavior of a system is defined by a $\mathcal{B}$-*coalgebra*, for an *endofunctor* $\mathfrak{B}$.

$\mathcal{M}=(M,\theta)$, where $M \xrightarrow{\theta} \mathfrak{B}M$

Bisimulation

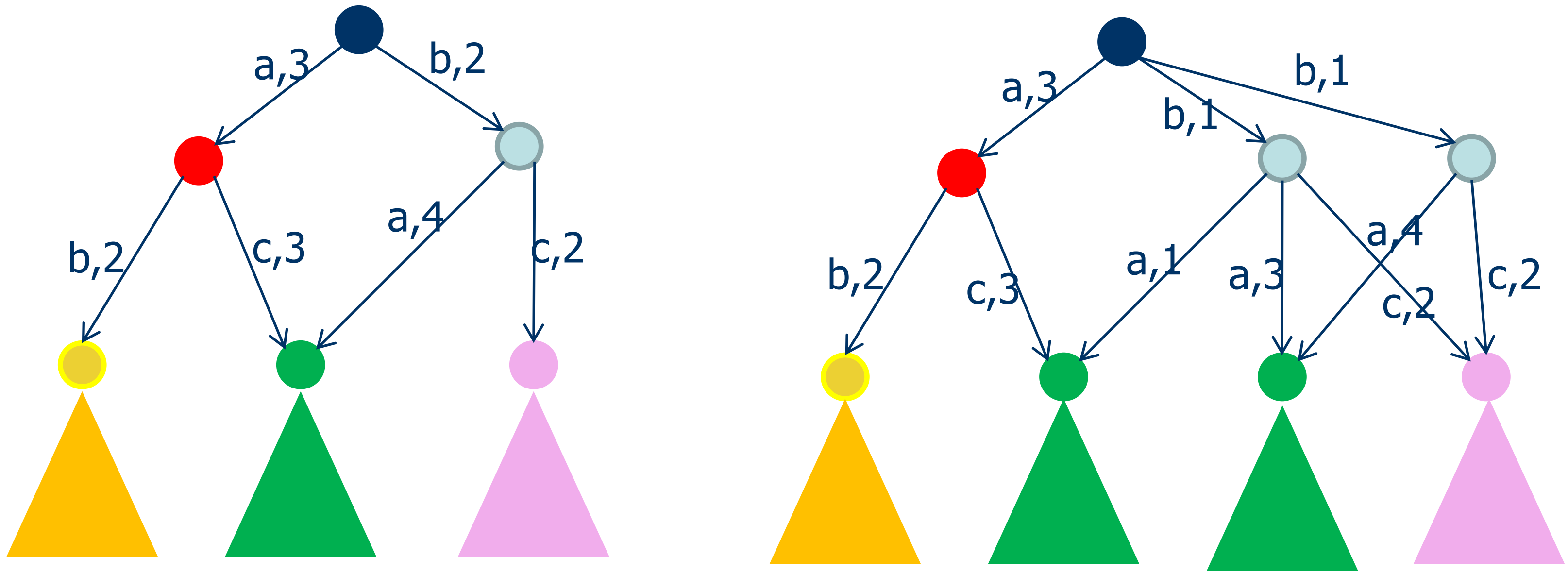
Given a **probabilistic/Markovian system** $\mathcal{M}=(M,\Sigma,\theta)$, a bisimulation relation is an equivalence relation $\sim\subseteq M\times M$ such that whenever $m_1\sim m_2$,

for arbitrary $S\in\Sigma(\sim)$ and $a\in A$

- If $m_1 \xrightarrow{a,p} S$, then $m_2 \xrightarrow{a,p} S$ and
- If $m_2 \xrightarrow{a,p} S$, then $m_1 \xrightarrow{a,p} S$.

K. G. Larsen and A. Skou. *Bisimulation through probabilistic testing*, 1991

P. Panangaden , *Labelled Markov Processes*, 2009.



Bisimulation

Given a **labelled transition system** $\mathcal{M}=(M,2^M,\theta)$, a bisimulation relation is an equivalence relation $\sim\subseteq M\times M$ such that whenever $m_1\sim m_2$, for arbitrary $a\in A$

- If $m_1 \xrightarrow{a} m'_1$, there exists $m'_2\in M$ such that $m_2 \xrightarrow{a} m'_2$ and $m'_1\sim m'_2$
- If $m_2 \xrightarrow{a} m'_2$, there exists $m'_1\in M$ such that $m_1 \xrightarrow{a} m'_1$ and $m'_1\sim m'_2$

Given a **probabilistic/Markovian system** $\mathcal{M}=(M,\Sigma,\theta)$, a bisimulation relation is an equivalence relation $\sim\subseteq M\times M$ such that whenever $m_1\sim m_2$,

for arbitrary $S\in\Sigma(\sim)$ and $a\in A$

- If $m_1 \xrightarrow{a,p} S$, then $m_2 \xrightarrow{a,p} S$ and
- If $m_2 \xrightarrow{a,p} S$, then $m_1 \xrightarrow{a,p} S$.

Alternatively:

Given a system $\mathcal{M}=(M,\Sigma,\theta)$, a bisimulation relation is an equivalence relation $\sim\subseteq M\times M$ such that whenever $m\sim m'$, for arbitrary $S\in\Sigma(\sim)$ and $a\in A$,

$$\theta_a(m)(S)=\theta_a(m')(S)$$

Focusing on modularity… algebraically

- **The coalgebraic structure**

Continuous Markov kernel
a tuple $\mathcal{M}=(M,\Sigma,\theta)$, where (M,Σ) is an analytic set and $\theta \in [\![M \rightarrow \Delta(M,\Sigma)]\!]^A$.

- **The algebraic structure**

Synchronization function: $* : A \times A \hookrightarrow A$ that is *commutative*: $a*b=b*a$.

$c=a*b$ is the result of synchronizing a and b.

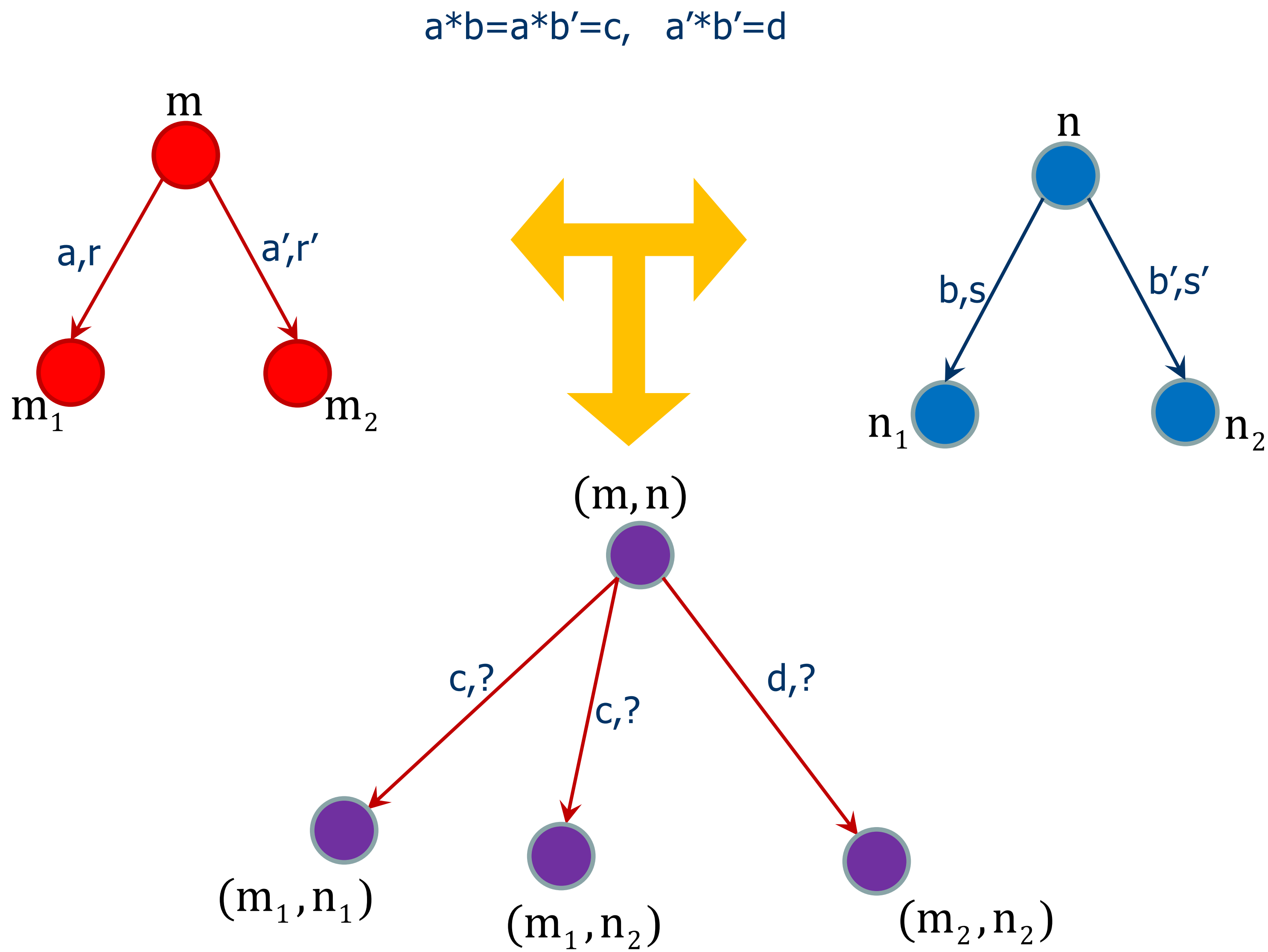
Examples:

CCS: $a*a'=\tau$, for a fixed $\tau \in A$;

CSP: $a*a=a$;

interleaving: $a*\delta=a$, for a reflexive $\delta \in A$.

Modular Markov Processes



Modular Markov Processes

- **The coalgebraic structure**

Continuous Markov kernel
a tuple $\mathcal{M}=(M,\Sigma,\theta)$, where (M,Σ) is an analytic set and $\theta\in[\![M \to \Delta(M,\Sigma)]\!]^A$.

- **The algebraic structure**

Synchronization function: $* : A\times A \hookrightarrow A$ (commutative).

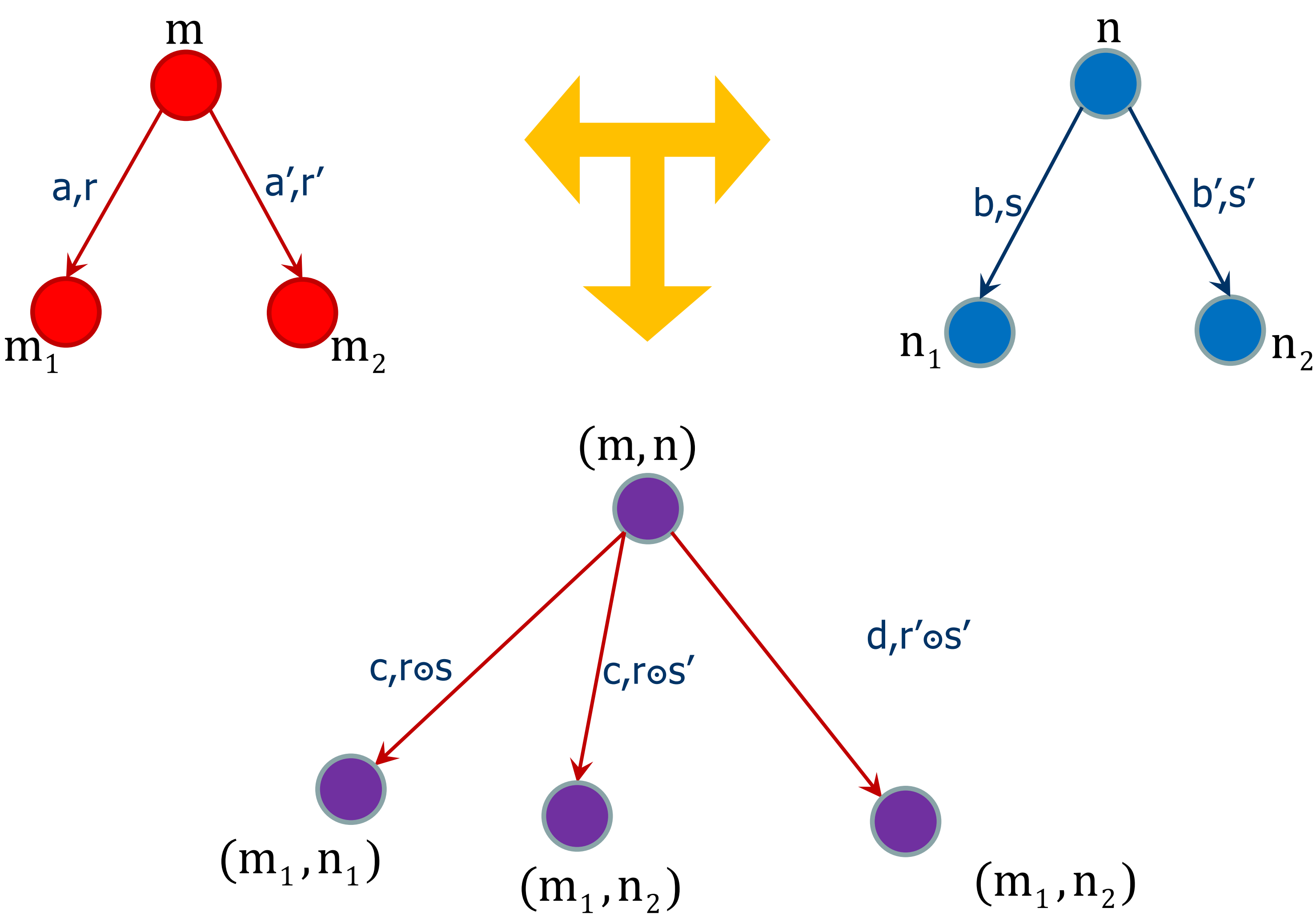
Rate function: $\odot : \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is
commutative: $r\odot s=s\odot r$,
associative: $(r\odot s)\odot t= r\odot(s\odot t)$,
bilinear: $r\odot(s+t)=(r\odot s)+(r\odot t)$, $(s+t)\odot r=(s\odot r)+(t\odot r)$,
continuous.

by synchronizing a with rate r and b with rate s, we obtain a*b with rate $r\odot s$.

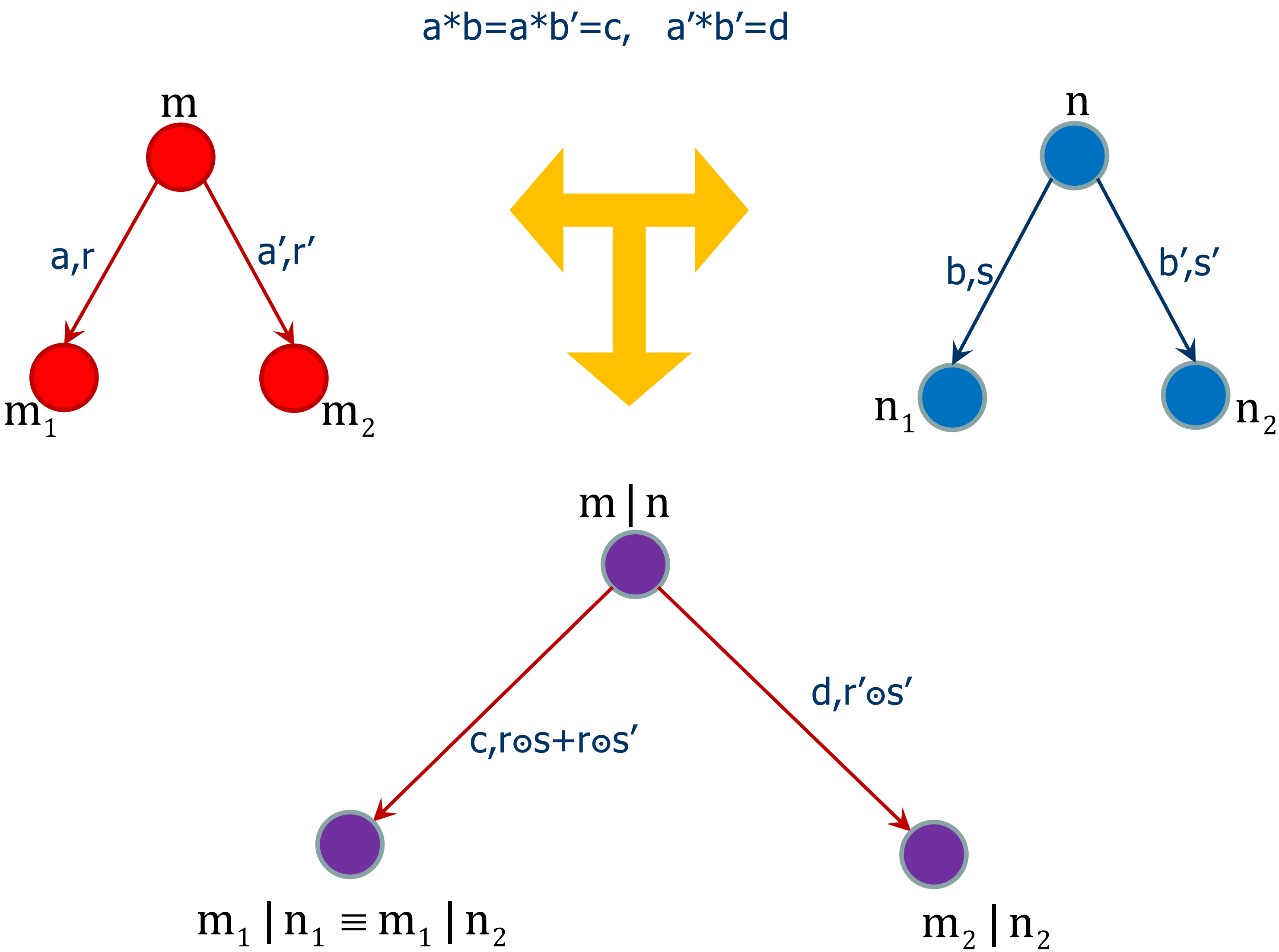
Examples:
The mass action law in Chemical Kinetics;
The minimal rate law in performance evaluation.

Modular Markov Processes

$a*b=a*b'=c, \quad a'*b'=d$



Modular Markov Processes



Modular Markov Processes

Continuous Markov kernel

a tuple $\mathcal{M}=(M,\Sigma,\theta)$, where (M,Σ) is an analytic set and $\theta \in [\![M \rightarrow \Delta(M,\Sigma)]\!]^A$.

Modular Markov kernel

a tuple $\mathcal{M}=(M,\Sigma,\theta,|,\equiv)$, where

- (M,Σ,θ) is a continuous Markov kernel
- $(M,|,\equiv)$ is a modular structure
    - $m|m' \equiv m'|m$, $(m|m')|m'' \equiv m|(m'|m'')$
    - $m \equiv m' \Rightarrow m|m'' \equiv m'|m''$
    - $\forall m$, $\exists!\ m_1,\ldots,m_k$ – atomic modules, $m \equiv m_1|\ldots|m_k$
- $(m',m'') \sim m'|m''$

Modular Markov process

a tuple $(\mathcal{M},m)$, where $\mathcal{M}=(M,\Sigma,\theta,|,\equiv)$ is a modular Markov kernel and $m \in M$

Structural Bisimulation

Modular Markov kernel

a tuple $\mathcal{M}=(M,\Sigma,\theta,|,\equiv)$, where

- (M,Σ,θ) is a continuous Markov kernel
- $(M,|,\equiv)$ is a modular structure
- $(m',m'') \sim m'|m''$

Structural bisimulation

Two modular Markov processes $(\mathcal{M},m)$ and $(\mathcal{M},n)$ are structural bisimilar, $m \cong n$, if

- $m \sim n$
- if $m \equiv m_1|\ldots|m_k$, then $n \equiv n_1|\ldots|n_k$ and $m_i \cong n_i$
- if $n \equiv n_1|\ldots|n_k$, then $m \equiv m_1|\ldots|m_k$ and $m_i \cong n_i$

Lemma: Structural bisimulation is a congruence w.r.t. the modular structure, i.e., if $m' \cong n'$ and $m'' \cong n''$, then $m'|m'' \cong n'|n''$.

Models and Properties

Models

e.g. coalgebras, algebras, bialgebras, transition systems, Markov processes, process algebras, Petri Nets, automata

- describe globally and exhaustively the systems; are not "authentic" logical structures (do not involve Boolean operators: ¬,∧,∨,→,T).
- behavioural/structural equivalence: when different systems are indistinguishable from a modeling perspective

Properties

e.g. modal logics, dynamic logics, Hennessy-Milner logic, temporal (probabilistic/stochastic) logics, equational and co-equational logics

- describe properties of systems in given states; can specify partial properties; use the Boolean operators.
- logical equivalence: when two systems or processes satisfy the same properties

The challenge: given a class of systems (bialgebras), define a logic for them such that

structural bisimulation=logical equivalence

Modular Markovian Logic

Syntax: MML(A)

$$f := T \mid \neg f \mid f_1 \wedge f_2$$

Semantics: Let $(m,\mathcal{M})$ be an arbitrary MMP with $\mathcal{M}=(M,\Sigma,\theta,|,\equiv)$.

$(m,\mathcal{M}) \models T$	always	
$(m,\mathcal{M}) \models \neg f$	iff	$(m,\mathcal{M}) \not\models f$
$(m,\mathcal{M}) \models f_1 \wedge f_2$	iff	$(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$

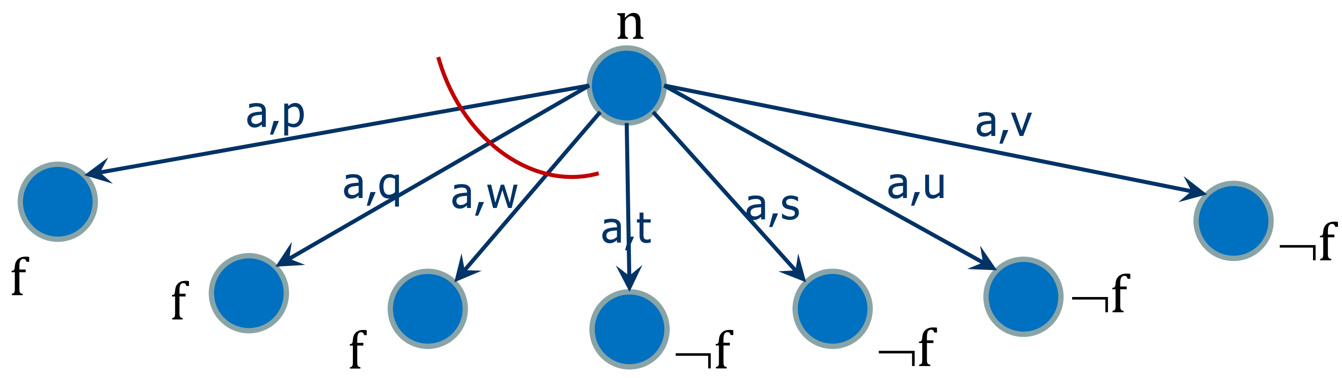
Modular Markovian Logic

Syntax: MML(A)

$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f$ $\qquad r \in \mathbb{Q}_+ \ a \in A$

Semantics: Let $(m,\mathcal{M})$ be an arbitrary MMP with $\mathcal{M}=(M,\Sigma,\theta,|,\equiv)$.

$(m,\mathcal{M}) \models T$	always
$(m,\mathcal{M}) \models \neg f$	iff $(m,\mathcal{M}) \not\models f$
$(m,\mathcal{M}) \models f_1 \wedge f_2$	iff $(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$
$(m,\mathcal{M}) \models L^a_r f$	iff $\theta_a(m)([f]) \geq r$, where $[f]=\{n \in M \mid (n,\mathcal{M}) \models f\}$



Modular Markovian Logic

Syntax: MML(A)

$$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f \mid f_1 | f_2 \qquad r \in \mathbb{Q}_+ \ a \in A$$

Semantics: Let $(m,\mathcal{M})$ be an arbitrary MMP with $\mathcal{M}=(M,\Sigma,\theta,|,\equiv)$.

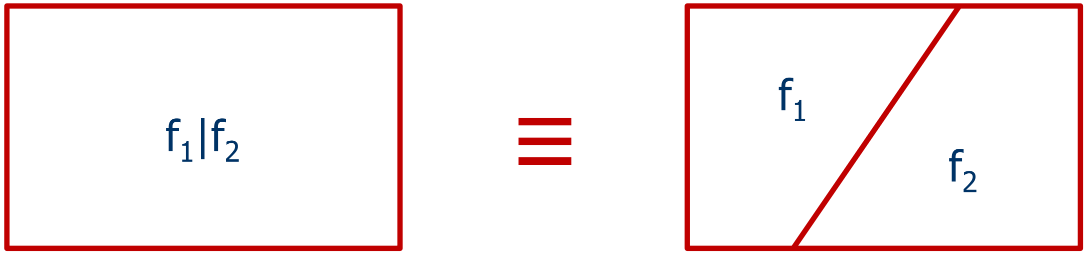
$(m,\mathcal{M}) \models T$ always

$(m,\mathcal{M}) \models \neg f$ iff $(m,\mathcal{M}) \not\models f$

$(m,\mathcal{M}) \models f_1 \wedge f_2$ iff $(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$

$(m,\mathcal{M}) \models L^a_r f$ iff $\theta_a(m)([f]) \geq r$, where $[f]=\{n \in M \mid (n,\mathcal{M}) \models f\}$

$(m,\mathcal{M}) \models f_1 | f_2$ iff $\exists$ n,k∈M, $m \equiv n|k$, $(n,\mathcal{M}) \models f_1$ and $(k,\mathcal{M}) \models f_2$



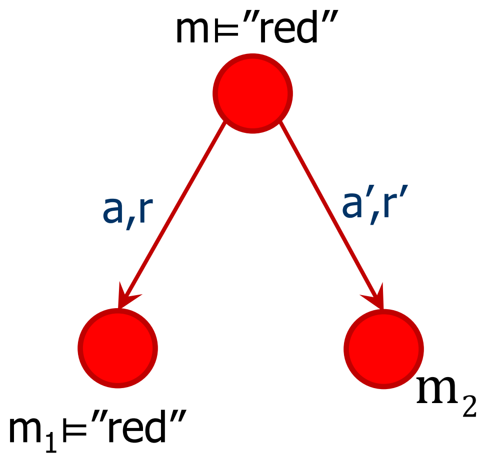
A. Urquhart, *Semantics for Relevant Logics*, 1972.
L. Cardelli, A. D. Gordon, *Anytime, Anywhere. Modal Logics for Mobile Ambients*, 2000.
J. C. Reynolds, *Separation Logic: A Logic for Shared Mutable Data Structures*, 2002.

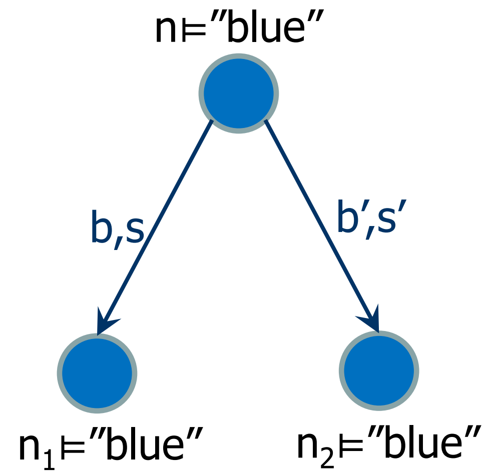
Modular Markovian Logic

$a*b = a*b' = c, \quad a'*b' = d$

$m \models L^a_r$ "red"
$m \models \neg L^b_s$ "blue"

$n \models L^b_s$ "blue"
$n \models \neg L^a_r$ "red"

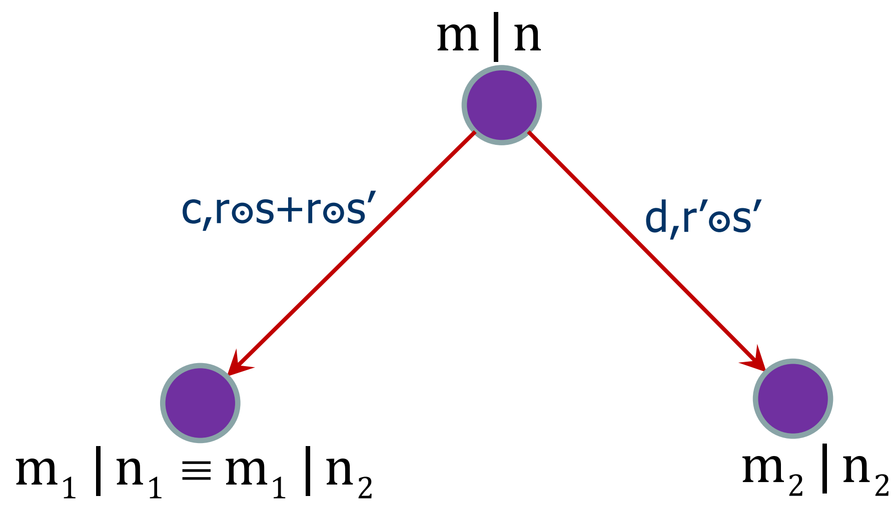




$m|n \models L^c_0$ "blue"
$m|n \models L^c_0$ "red"

$m|n \models L^a_r$ "red" $|$ L^b_s "blue"
$m|n \models \neg L^a_r$ "red" $|$ $\neg L^b_s$ "blue"

$m|n \models L^c_{(r\odot s+r\odot s')}$ "purple"
$m|n \models L^d_{r'\odot s'}$ "purple"

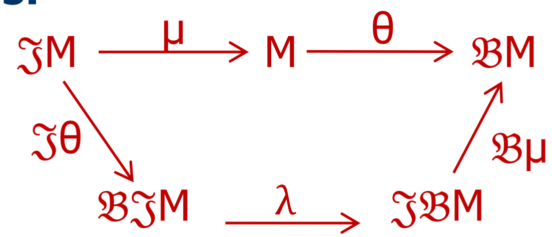


Modular Markovian Logic

Syntax: MML(A)

$f := T \mid \neg f \mid f_1 \wedge f_2 \mid L^a_r f \mid f_1 | f_2$, $r \in \mathbb{Q}_+$ $a \in A$

Models:



Semantics: Let $(m,\mathcal{M})$ be an arbitrary MMP.

$(m,\mathcal{M}) \models T$ always

$(m,\mathcal{M}) \models \neg f$ iff $(m,\mathcal{M}) \not\models f$

$(m,\mathcal{M}) \models f_1 \wedge f_2$ iff $(m,\mathcal{M}) \models f_1$ and $(m,\mathcal{M}) \models f_2$

$(m,\mathcal{M}) \models L^a_r f$ iff $\theta_a(m)([f]) \geq r$, where $[f] = \{n \in M \mid (n,\mathcal{M}) \models f\}$

$(m,\mathcal{M}) \models f_1 | f_2$ iff $\exists\, n,k \in M$, $m \equiv n|k$, $(n,\mathcal{M}) \models f_1$ and $(k,\mathcal{M}) \models f_2$

Modalities:

- (X,R), $R \subseteq X \times X$

$(X,R,x) \models \Diamond f$ iff $\exists\, x' \in X$, $(x,x') \in R$ and $(X,R,x') \models f$

- (X,R'), $R' \subseteq X \times X \times X$

$(X,R',x) \models \Delta(f_1,f_2)$ iff $\exists\, x',x'' \in X$, $(x,x',x'') \in R'$ and $(X,R',x') \models f_1$, $(X,R',x'') \models f_2$

Axiomatization of Modular Markovian Logic

(A1) $\vdash L^a_0 f$

(A2) $\vdash L^a_{r+s} f \rightarrow L^a_r f$

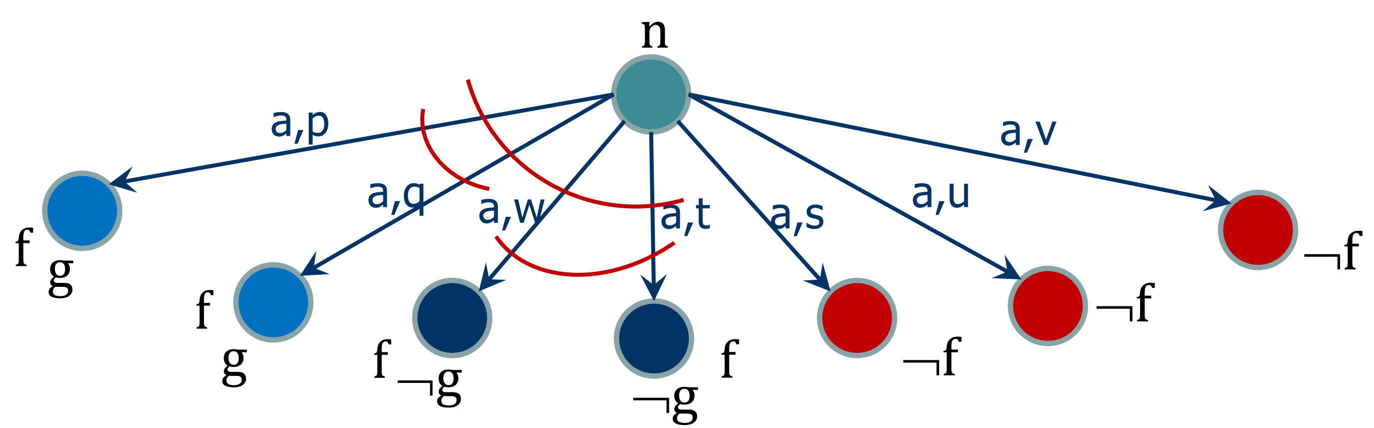
(A3) $\vdash L^a_r(f \wedge g) \wedge L^a_s(f \wedge \neg g) \rightarrow L^a_{r+s} f$

(A4) $\vdash \neg L^a_r(f \wedge g) \wedge \neg L^a_s(f \wedge \neg g) \rightarrow \neg L^a_{r+s} f$

(R1) If $\vdash f \rightarrow g$, then $\vdash L^a_r f \rightarrow L^a_r g$

(R2) If $\forall r<s$, $\vdash f \rightarrow L^a_r g$, then $\vdash f \rightarrow L^a_s g$

(R3) If $\forall r>s$, $\vdash f \rightarrow L^a_r g$, then $\vdash f \rightarrow \neg T$



L. Cardelli, K. G. Larsen, R. Mardare, *Continuous Markovian Logic - From Complete Axiomatization to the Metric Space of Formulas*, CSL 2011.

Axiomatization of Modular Markovian Logic

(A5) $\vdash f|g \rightarrow g|f$

(A6) $\vdash f|(g|h) \rightarrow (f|g)|h$

(A7) $\vdash f|\neg T \rightarrow \neg T$

(A8) $\vdash f|(g \vee h) \rightarrow f|g \vee f|h$

(R4) If $\vdash f \rightarrow g$, then $\vdash f|h \rightarrow g|h$

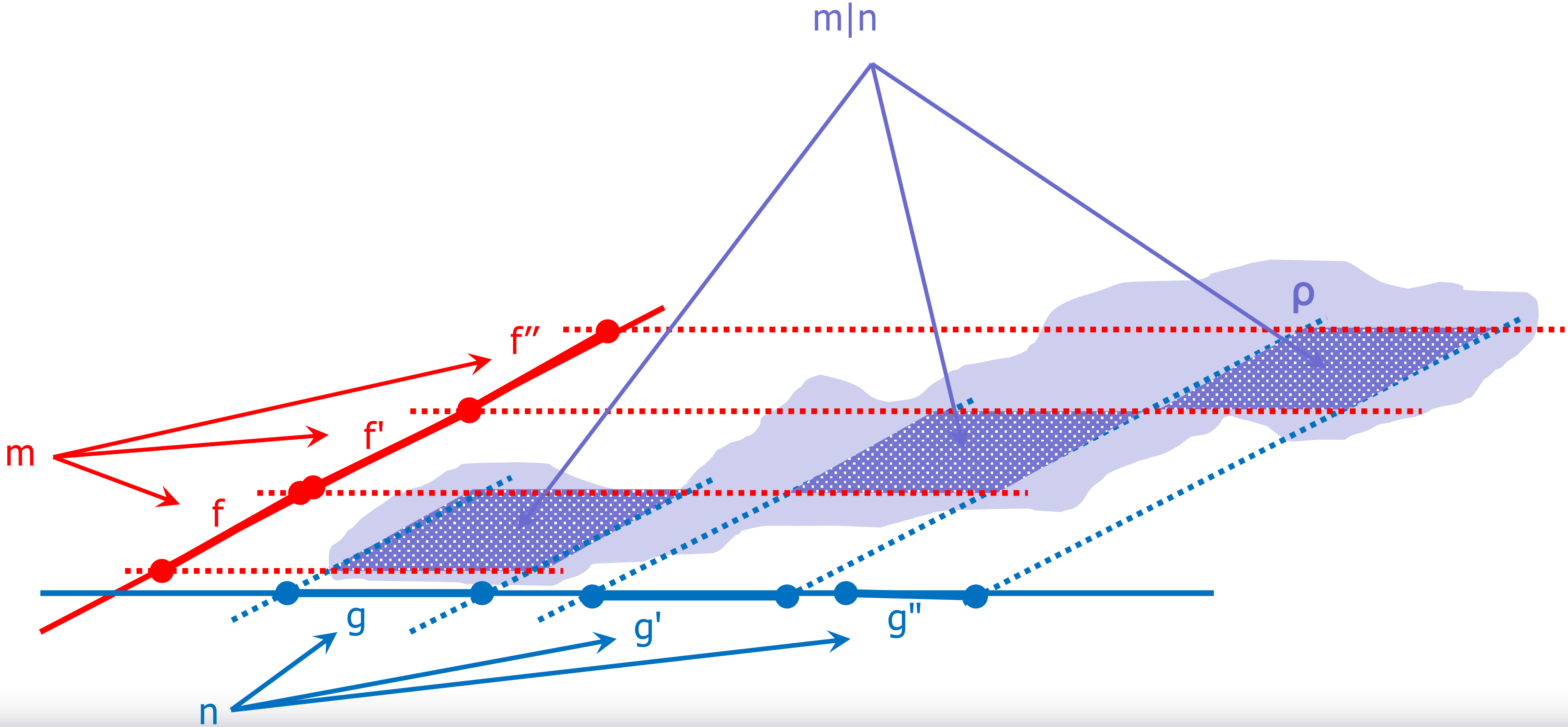
(R5) If $\vdash f \rightarrow f|T$, then $\vdash f \rightarrow \neg T$

R. Mardare, A. Policriti, *A Complete Axiomatic System for Process-based Spatial Logic*, 2008.

Axiomatization of Modular Markovian Logic

(R8): If K is finite and $\vdash (\boxed{+}_{k\in K} \phi_k^0) \wedge (\boxed{+}_{k\in K} \phi_k^1) \wedge (\bigvee_{k\in K} \phi_k^0|\phi_k^1 \to \rho)$, then

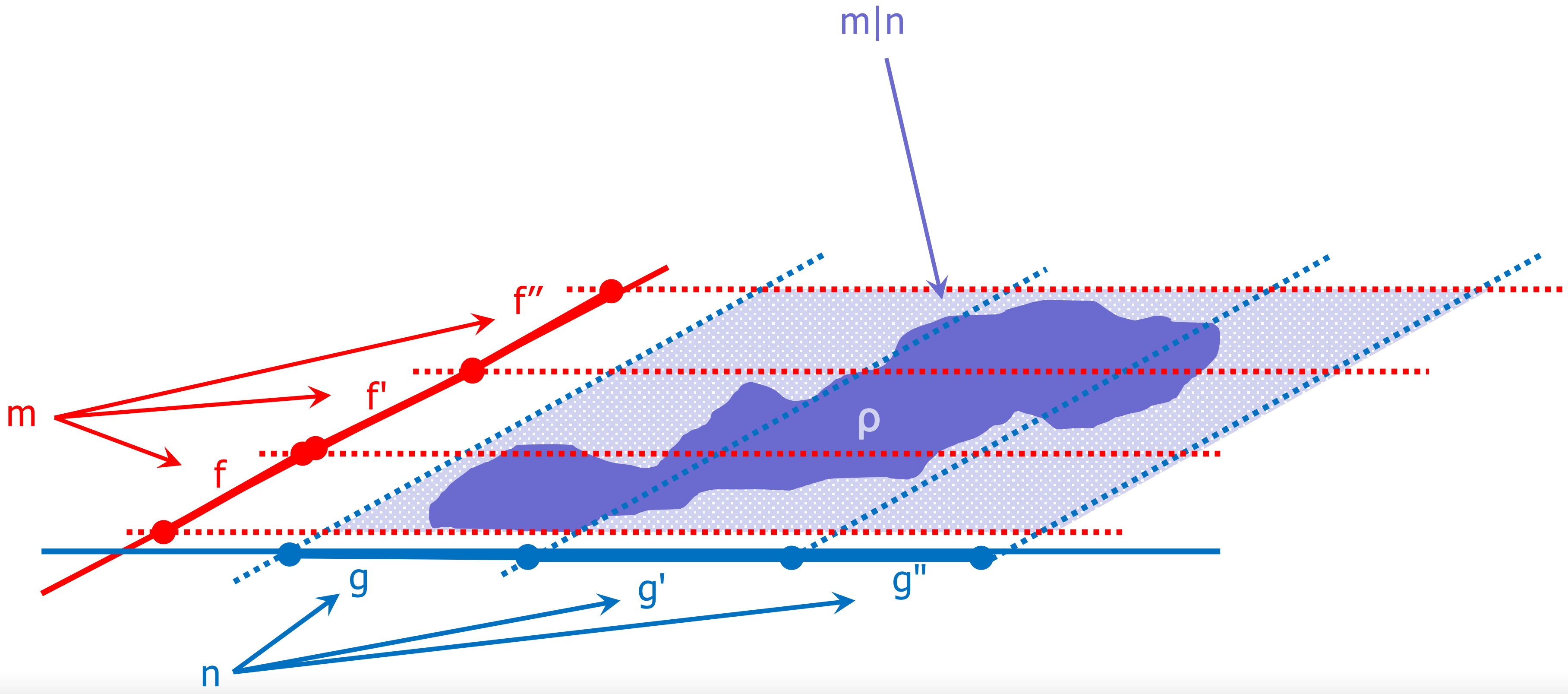
$$\vdash \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} L^{b_i}_{(r_k^{i,j})} \phi_k^j \right) \Big| \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} L^{c_i}_{(s_k^{i,j})} \phi_k^{1-j} \right) \to L^a_{(r_K^I \bullet s_K^I)} \rho$$



Axiomatization of Modular Markovian Logic

(R9): If K is finite and $\vdash (\boxed{+}_{k\in K}\, \phi_k^0) \wedge (\boxed{+}_{k\in K}\, \phi_k^1) \wedge (\rho \rightarrow \bigvee_{k\in K} \phi_k^0 | \phi_k^1)$, then

$$\vdash \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} \neg L^{b_i}_{(r_k^{i,j})} \phi_k^j \right) \Bigg| \left(\bigwedge_{k\in K}^{i\in I_a} \bigwedge_{j=0,1} \neg L^{c_i}_{(s_k^{i,j})} \phi_k^{1-j} \right) \rightarrow \neg L^a_{(r_K^I \bullet s_K^I)} \rho$$



Axiomatization of Modular Markovian Logic

Theorems:

(T1) $\vdash L^a_r T \mid L^b_s T \rightarrow L^{a*b}_{r \odot s} T|T$

(T2) $\vdash (L^a_r f \wedge L^b_s g) \mid (L^a_{r'} f \wedge L^b_{s'} g) \rightarrow L^{a*b}_{(r \odot s' + s \odot r')} f|g$

$$\frac{P_1 \vDash f_1,\ P_2 \vDash f_2,\ \ldots,\ P_k \vDash f_k}{P_1|P_2|\ldots|P_k \vDash f}\quad C(f, f_1, f_2, \ldots, f_k)$$

$$\frac{P_1 \vDash L^a_r T,\quad P_2 \vDash L^b_s T}{P_1|P_2 \vDash L^{a*b}_{r \odot s} T|T}$$

$$\frac{P_1 \vDash L^a_r f \wedge L^b_s g,\quad P_2 \vDash L^a_{r'} f \wedge L^b_{s'} g}{P_1|P_2 \vDash L^{a*b}_{(r \odot s' + s \odot r')} f|g}$$

$$\frac{m \vDash L^a_r\text{"red"} \wedge L^b_0\text{"blue"},\quad n \vDash L^a_0\text{"red"} \wedge L^b_s\text{"blue"}}{m|n \vDash L^{a*b}_{r \odot s}\text{"purple"}}\quad \vDash \text{"red"}|\text{"blue"} \rightarrow \text{"purple"}$$

Metaproperties

Two modular Markov processes ($\mathcal{M}$,m) and ($\mathcal{M}$,n) are structural bisimilar, $m \cong n$, if

- $m \sim n$
- if $m \equiv m_1|\ldots|m_k$, then $n \equiv n_1|\ldots|n_k$ and $m_i \cong n_i$
- if $n \equiv n_1|\ldots|n_k$, then $m \equiv m_1|\ldots|m_k$ and $m_i \cong n_i$

Metatheorem [Logical characterisation of Structural Bisimulation]:

For arbitrary modular Markov processes ($m,\mathcal{M}$) and ($n,\mathcal{H}$),

$$(m,\mathcal{M}) \cong (n,\mathcal{H}) \quad \text{iff} \quad [\forall f \in MML(A),\ (m,\mathcal{M}) \models f \ \text{ iff } \ (n,\mathcal{H}) \models f\ ].$$

Corollary:

For arbitrary modular Markov processes ($m,\mathcal{M}$) and ($n,\mathcal{H}$),

$$\text{If } [\forall f \in MML(A),\ (m,\mathcal{M}) \models f \ \text{ iff } \ (n,\mathcal{H}) \models f] \text{ then } (m,\mathcal{M}) \sim (n,\mathcal{H}).$$

Metaproperties

If for any modular Markov process $(m,\mathcal{M})$, $(m,\mathcal{M})\vDash f$, we write $\vDash f$.
We write $\vdash f$ if f is either an axiom or it can be proved from the axioms of MML(A).

Metatheorem 1 [Small model property]:

If f is consistent, there exists a modular Markov process that satisfies f.

Moreover, its support is finite and bound by the dimension of f.

Metatheorem 2 [Soundness & Weak Completeness]:

The axiomatic system of MML(A) is sound-complete w.r.t. the Markovian semantics,

$$\vdash f \text{ iff } \vDash f.$$

Summary

- We have developed a class of models for **continuous-time and continuous-space Markov processes**.
- Our Markov processes are **compositional** and encode a fairly **general notion of synchronization/communication**.
- The concept of **stochastic bisimulation** generalizes the one of probabilistic bisimulation.
- Stochastic bisimulation is **“invariant” to parallel composition**.
- We have introduced the **Modular Markovian Logic** (MML) for Markov processes.
- MML **characterizes the stochastic bisimulation** and can encode modular properties.
- MML enjoys the **small model property**.
- We have identified a sound and complete **Hilbert-style axiomatic system** for MML.